Bratteli-Vershik Models from Branching Rauzy Induction

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Interval Exchange Transformations

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Definition (interval exchange transformation)

Let $\mathcal{A} = \{a_1 < \cdots < a_n\}$ be an alphabet of $n \geq 1$ letters, and fix points $\ell = z_0 < z_1 < \cdots < z_n = r$. Consider the partition of $I = [\ell, r)$ into the subintervals $I_{a_i} = [z_{i-1}, z_i)$, $1 \leq i \leq n$. Given $\pi \in S_{\mathcal{A}}$, the associated *n-interval exchange transformation* is the map $T : I \to I$ acting on each sub-interval by a translation

$$T(z) = z + \tau_a,\ z \in I_a, \qquad \tau_a = \sum_{\pi^{-1}(b) < \pi^{-1}(a)} |I_b| - \sum_{b<a} |I_b|.$$

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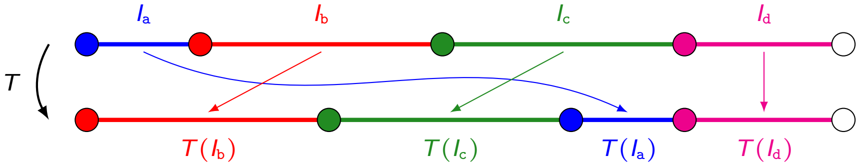

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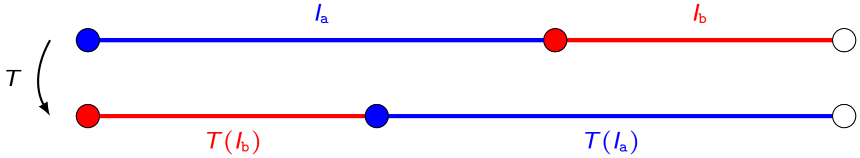

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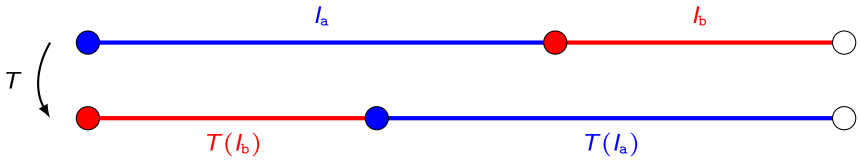

Definition (Formal discontinuity points)

Let (I, T) be an interval exchange transformation with a partition $\mathcal{P} = \{I_{a_1}, \ldots, I_{a_n}\}$, where $I_{a_i} = [z_{i-1}, z_i)$ and $\ell = z_0 < z_1 < \cdots < z_n = r$. The set of *formal discontinuity points* of T, denoted by $\mathcal{D}(T)$, is the set of points $z_1, \ldots, z_{n-1}$.

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All regular interval exchanges are minimal, but the converse is false.

From Interval Exchanges to Cantor Spaces

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$$X = ([\ell, r] \setminus \mathcal{O}(T)) \bigcup \{z^-, z^+ \mid z \in \mathcal{O}(T)\},$$

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Definition (Cantor version of an interval exchange)

Let (I, T) be a minimal interval exchange with X as defined previously. Define the map $\phi : X \to X$ by $\phi(x) = T(x)$ for $x \in X \setminus \{z^-, z^+ \mid z \in \mathcal{O}(T)\}$, $\phi(x^-) = (T(x))^-$, $\phi(x^+) = (T(x))^+$ for $x \in \mathcal{O}(T)$.
The dynamical system (X, ϕ) is called the *Cantor version of* T.

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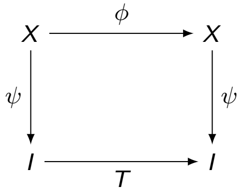

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the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X \\ \psi \downarrow & & \downarrow \psi \\ I & \xrightarrow[\ T\ ]{} & I \end{array}$$

Similarly, there is an inclusion map $\iota : I \to X$ defined by $\iota(x) = x$ when $x \in I \setminus \mathcal{O}(T)$, and $\iota(x) = x^+$ when $x \in \mathcal{O}(T)$ such that $(\iota \circ T)(x) = (\phi \circ \iota)(x)$ for all $x \in I$.

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Let (I, T) be a regular interval exchange and let (X, ϕ) be its Cantor version. If $D = [a, b)$ is a (branching) Rauzy induction domain for T, then the corresponding clopen set

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Thus, Cantorization and induction commute.

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Definition (Partition in towers)

Let (X, T) be an invertible Cantor minimal system. Let $\{I_1, \ldots, I_n\}$ be a collection of pairwise disjoint subsets of X, and let $\{h_1, \ldots, h_n\}$ be a set of n positive integers. If the collection $\{T^i(I_j) \mid 0 \leq i \leq h_j, 1 \leq j \leq n\}$ is a partition of X, it is called a *partition in towers* of (X, T).

The intervals $I_1, \ldots, I_n$ are called tower bases, the integers $h_1, \ldots, h_n$ are called tower heights, and the set $\{T^i(I_j) \mid 0 \leq i \leq h_j\}$ is called the j^{th} tower.

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The union $I_1 \cup \cdots \cup I_n$ is called the *basis* of the partition in towers.

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Definition (Nested sequence of partitions)

A sequence of partitions in towers $(\mathcal{B})_{n\geq 0}$ is said to be nested if $\mathcal{B}_{n+1}$ is nested in $\mathcal{B}_n$ for all $n \geq 0$.

Partitions in Towers from the Fibonacci IET

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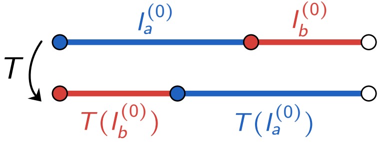

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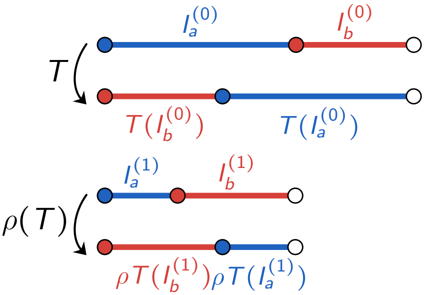

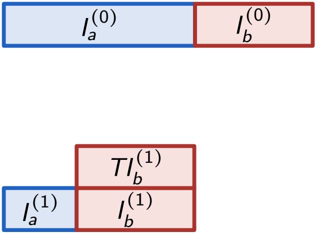

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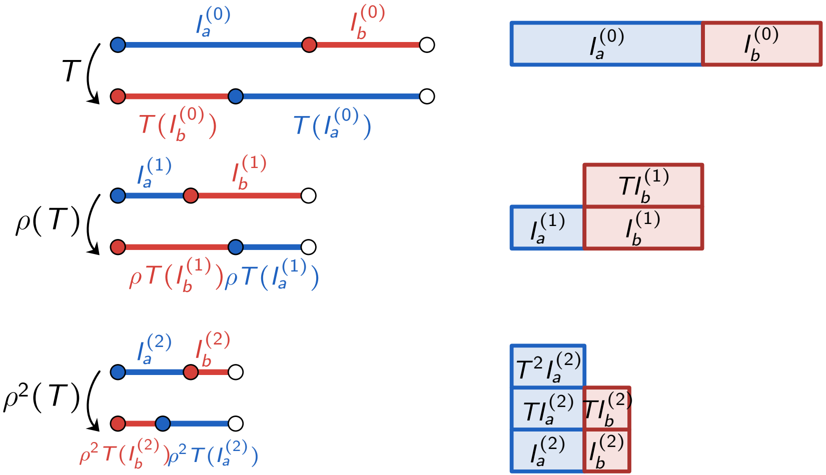

From Towers to Bratteli Diagrams

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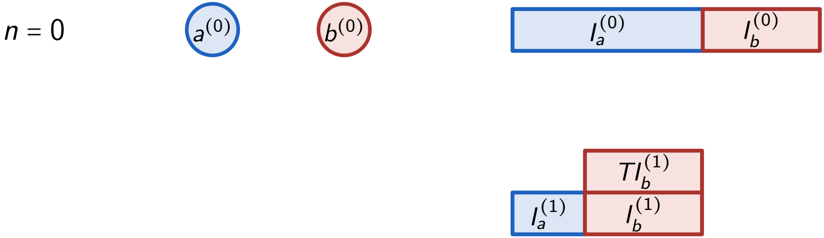

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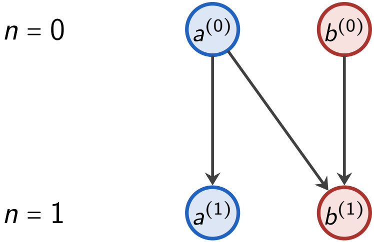

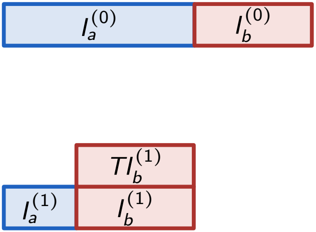

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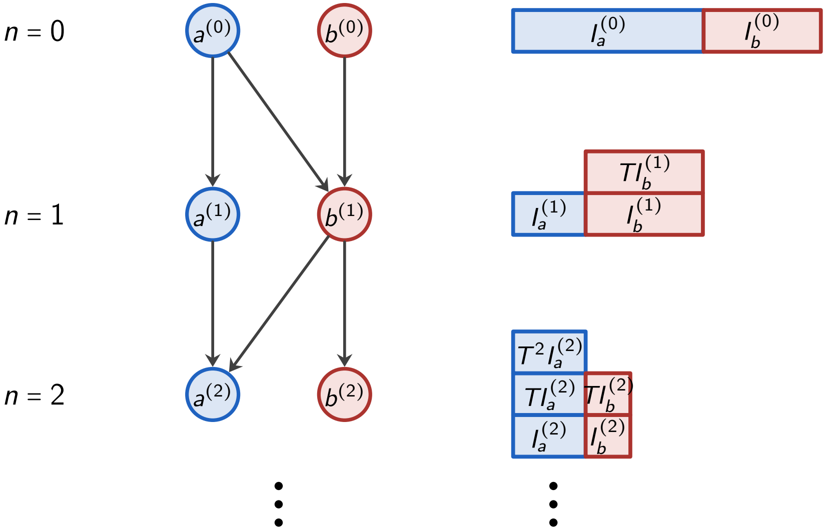
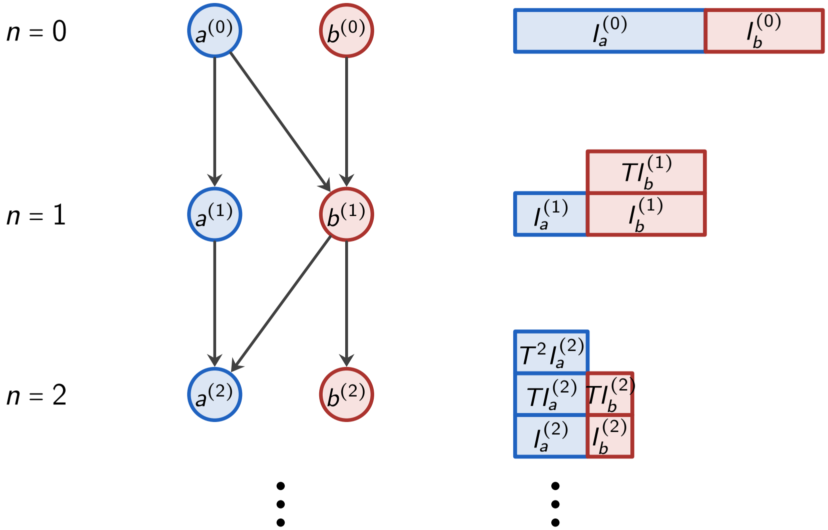


Ordering the Diagram



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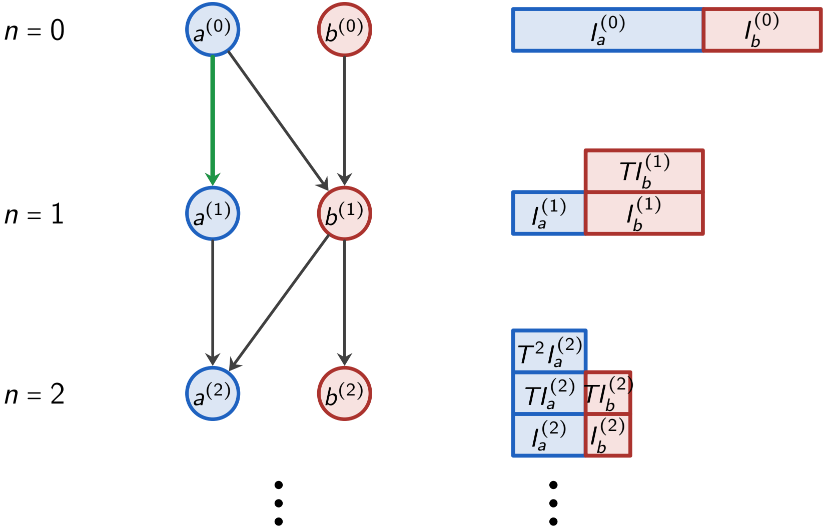

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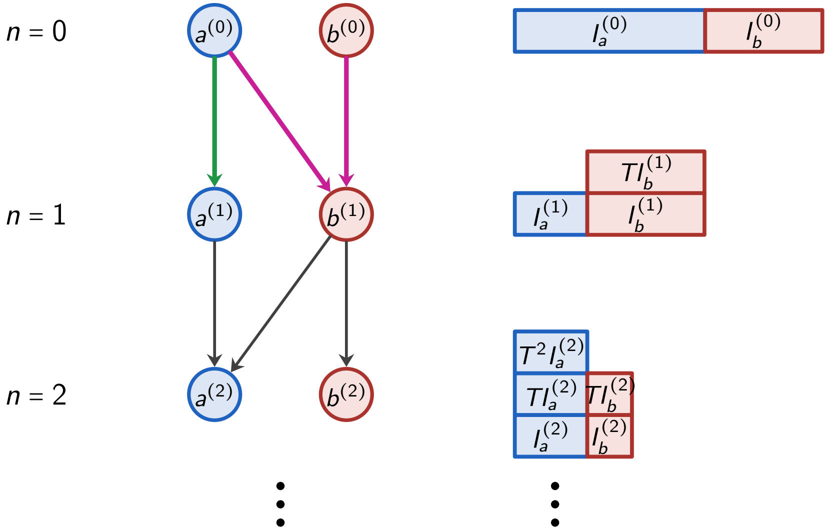

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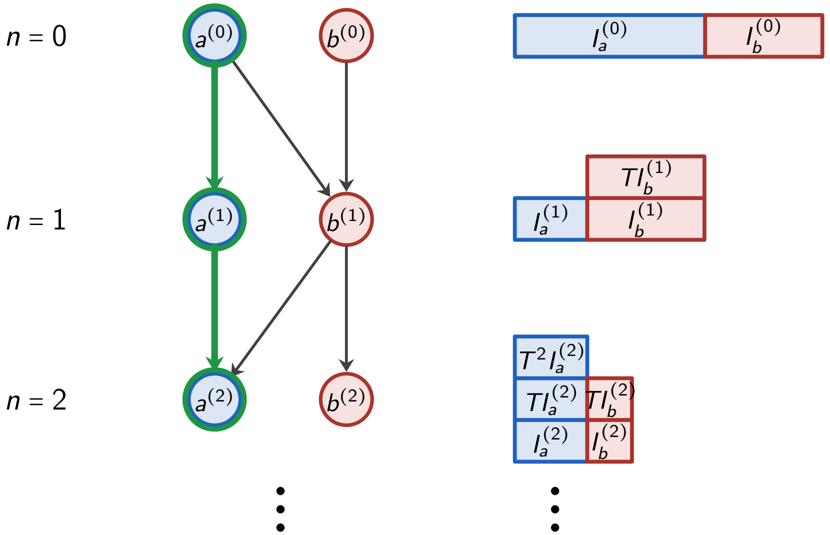

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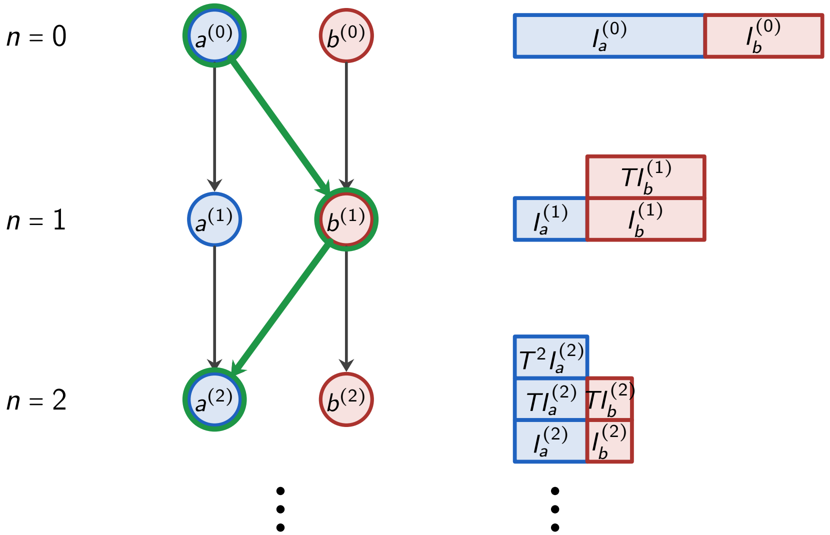

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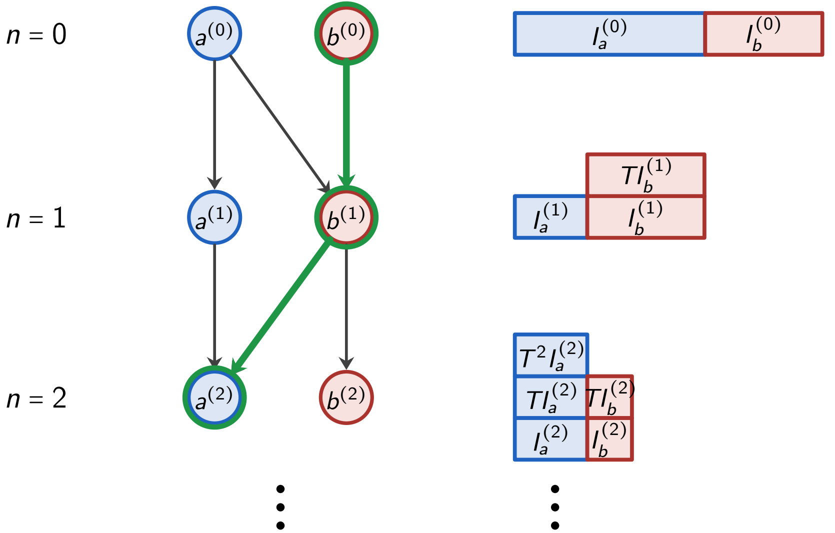

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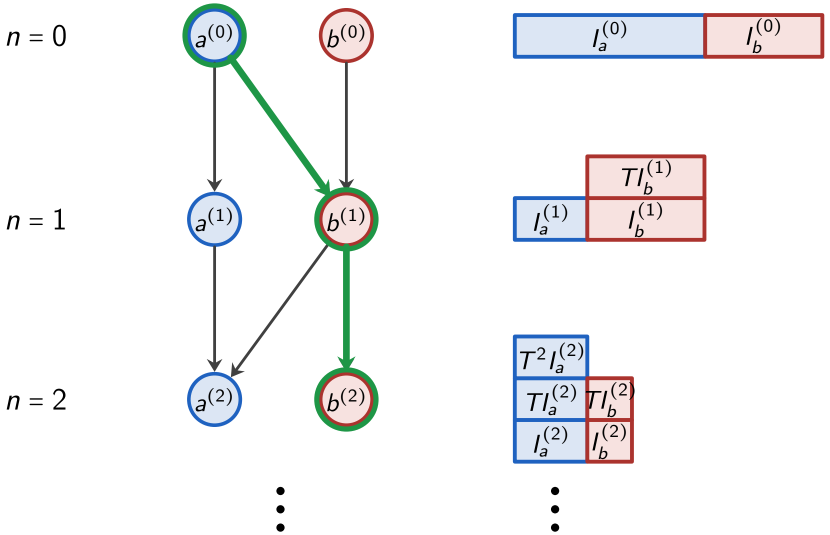

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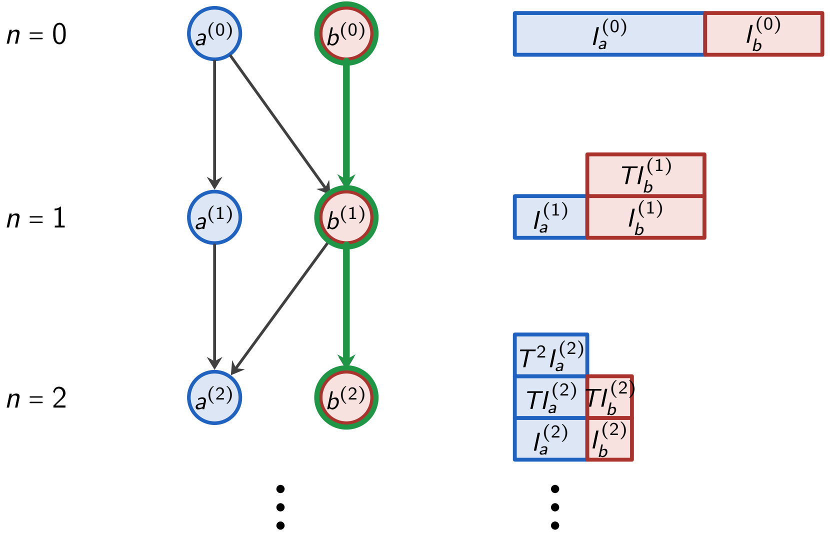

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Definition (Bratteli diagram)

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Thus, nested tower partitions naturally produce ordered Bratteli diagrams.

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Let M_n be the matrix whose rows are indexed by vertices at level n and whose columns are indexed by vertices at level $n-1$.

The entry $(M_n)_{v,w}$ counts the number of edges from w to v.

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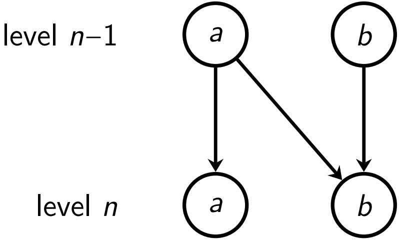
Products of incidence matrices count paths across several levels.

In particular, if sufficiently long products have all entries positive, then the diagram is *simple*.

The Incidence Matrix of One Level



$$\begin{array}{cc} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array}$$

The Path Space of a Bratteli Diagram

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Let $\mathcal{B} = (V, E)$ be a Bratteli diagram. The *infinite path space* of $\mathcal{B}$ is the set

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The infinite path space of a Bratteli diagram is compact, and the following sets form a clopen basis for it.

Definition (Path cylinder set)

A path cylinder set is a set $[e_1, e_2, \ldots, e_n]_E = \{(f_1, f_2, \ldots) \in X_E \,|\, f_i = e_i, \, 1 \leq i \leq n\}$.

Maximal and Minimal Paths

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Definition (Maximal/minimal path)

Let X_E be the infinite path space for an ordered Bratteli diagram and let $(e_1, e_2, \ldots e_i, \ldots) \in X_E$. The path $(e_1, e_2, \ldots, e_i, \ldots)$ is said to be *maximal* (resp. *minimal*) if each e_i is maximal (resp. minimal) on level i. We denote the set of such paths by $X_E^{\max}$ (resp. $X_E^{\min}$).

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Definition (Properly ordered Bratteli diagram)

Given an ordered Bratteli diagram $\mathcal{B} = (V, E, \leq)$ and its associated infinite path space, X_E, $\mathcal{B}$ is said to be *properly ordered* if it is simple and the sets $X_E^{\max}$ and $X_E^{\min}$ are singletons.

The Vershik Map

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Definition (Successor path in reverse lexicographic order)

Let $\mathcal{B} = (V, E, \leq)$ be an ordered Bratteli diagram, and let $(e_1, e_2, \ldots, e_i, \ldots) \in X_E$, where e_i is the first non-maximal edge in the path. The *successor path* of $(e_1, e_2, \ldots, e_i, \ldots)$ is the path $(f_1, f_2, \ldots, f_{i-1}, f_i, e_{i+1}, \ldots)$ obtained by setting e_i to its successor, f_i, on level i, then taking each f_{i-1} to be minimal (up to level-wise source-range agreement) for levels $1 \leq j \leq i-1$. The successor is undefined if the path is in $X_E^{\max}$.

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Definition (Vershik map)

Let X_E be the path space of a properly ordered Bratteli diagram $\mathcal{B}$ and let $x \in X_E$. The *Vershik map* $\varphi_E : X_E \to X_E$ associated with $\mathcal{B}$ is the map defined by $\varphi(x) = \text{successor}(x)$ if $x \notin X_E^{\max}$, and $\varphi(x) = x_{\min}$ if $x \in X_E^{\max}$, where $x_{\min}$ is the single path in $X_E^{\min}$.

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Can we relate the behavior of the Vershik map on the Bratteli diagram to the dynamical system from which the Brattteli diagram was generated ?

Refining Sequences of Partitions in Towers

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A sequence $(\mathfrak{P}_n)_{n\geq 0}$ of partitions in towers for a system (X, T) is said to be *refining* if

1. For the sequence of bases $(B_n)_{n\geq 0}$ of $(\mathfrak{P}_n)_{n\geq 0}$, $\bigcap_{n\geq 0} B_n$ is a single point.
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Theorem (Herman, Putnam, Skau)

Let $(\mathfrak{P}_n)_{n\geq 0}$ be a refining sequence of partitions in towers for a minimal invertible Cantor system (X, T). Then $(X, T) \cong (X_E, \varphi_E)$, where X_E is the path space of the ordered Bratteli diagram arising from $(\mathfrak{P}_n)_{n\geq 0}$.

The Gjerde-Johansen Construction

Theorem (Gjerde, Johansen (2002))

Let T be a minimal IET and let ϕ be its Cantor version. There exists a simple ordered Diagram (obtained via an acceleration of right cuts) upon which the Vershik map is conjugate to ϕ.

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Why Only Cut on the Right?

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Proposition (Dolce, H. 2026+)

Let (I, T) be a regular IET and (X, ϕ) and $\chi \in \{\lambda, \rho\}^\omega$. The sequence of partitions in towers associated with χ and ϕ is nesting and generates the topology of X.

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Proposition (Dolce, H. 2026+)

Let (I, T) and (X, ϕ) be as above. There exist countably many paths in $\chi \in \{\lambda, \rho\}^\omega$ for which the sequence of bases of the associated partitions in towers has an intersection with cardinality 2.

Fibonacci Revisited

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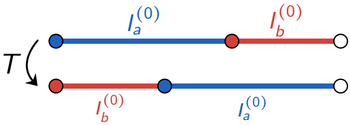
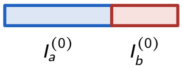
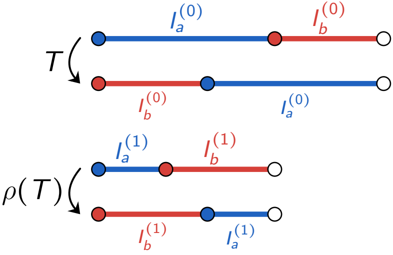
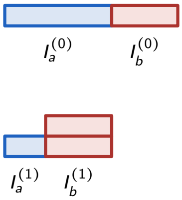




Fibonacci Revisited





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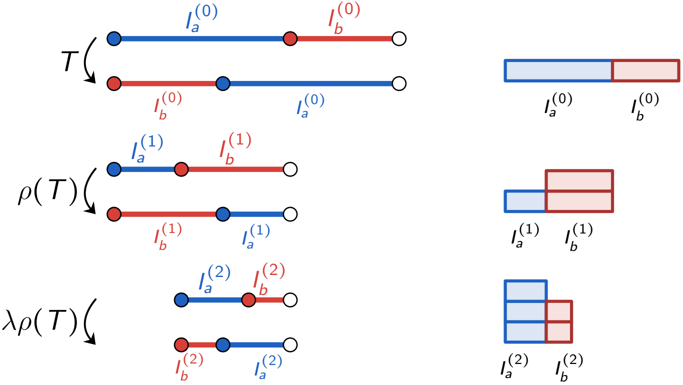

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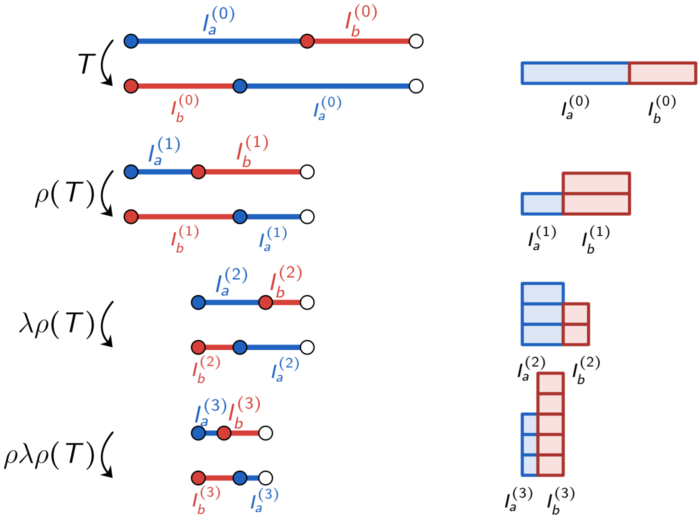

Obtaining a Conjugate Map Anyway

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Theorem (Dolce, H. 2026+)

Let (I, T) be a regular interval exchange and (X, ϕ) be its Cantorized version. For each $\chi \in \{\lambda, \rho\}^\omega$, there exists a homeomorphism ϕ_E on the path space X_E of the Bratteli diagram associated with ϕ and χ such that $(X_E, \phi_E) \cong (X, \phi)$. Furthermore, ϕ_E agrees with the successor map on paths $X_E \setminus X_E^{\max}$.

Merci Beaucoup !