Dynamics on an Interval: Interval Exchange Transformations

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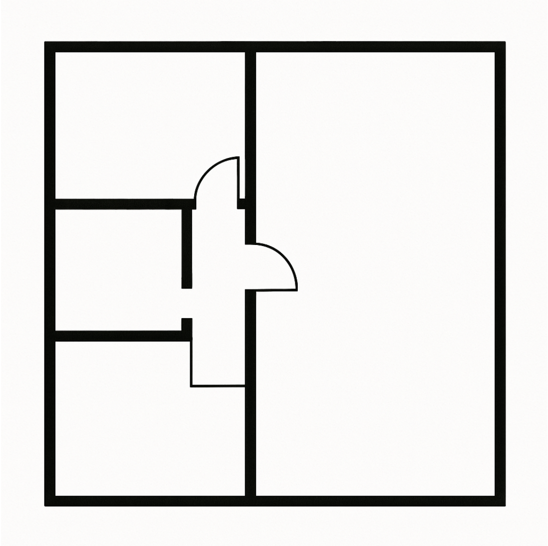
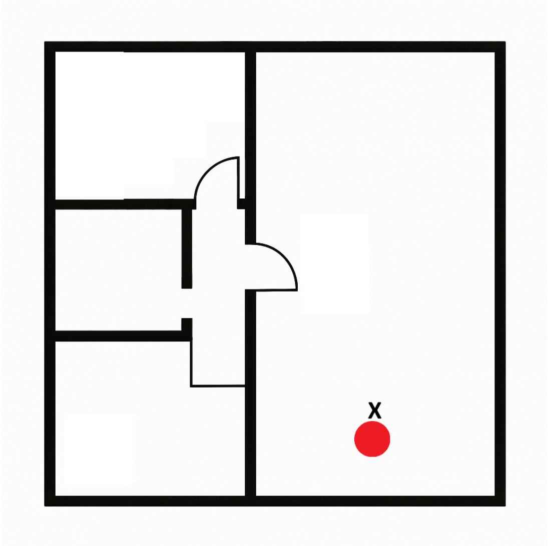
Motivating Question

What does it mean mathematically to evolve a space ? Our answer to this question determines how we describe many systems, from the spread of disease to black holes.

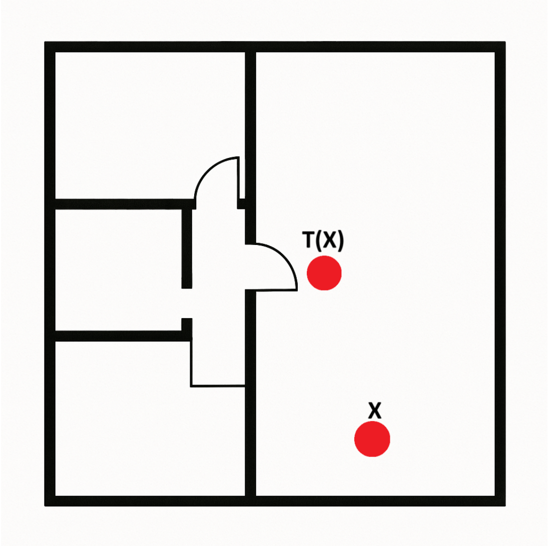
A House



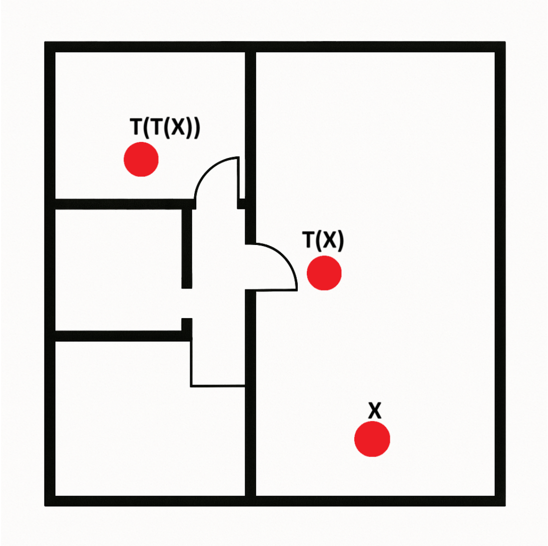
One Step



Two Step



Red Step



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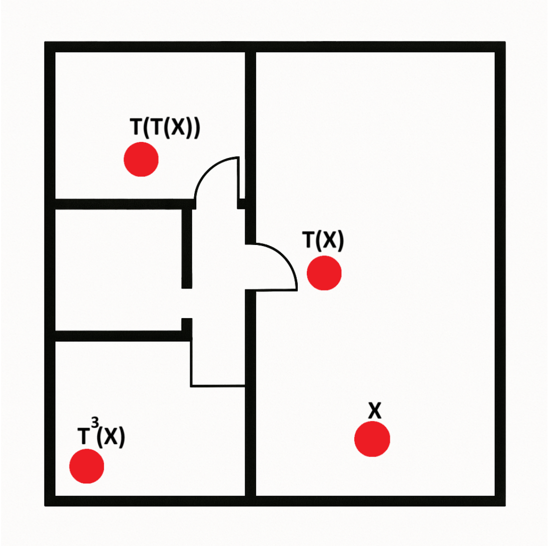

What Is a Dynamical System?

We have mentioned moving a point through space. Let us give this a mathematical framework.

Definition (Informal)

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Notation

For $n \in \mathbb{N}_0$, we denote n compositions of T with itself by T^n. By convention, T^0 denotes the identity on X.

More About T

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As a consequence of the nice notation we have chosen, we can now manipulate expressions like $T^m T^n$ as we would exponents, keeping the definitions in mind.

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Definition

Let (X, T) be a dynamical system. Let $x \in X$. We define the *positive* orbit of x to be the set $\mathcal{O}_+(x) = \{T^n(x) : n \in \mathbb{N}_0\}$. When T is invertible, we define the orbit of x to be the set $\mathcal{O}(x) = \{T^n(x) : n \in \mathbb{Z}\}$.

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Note : the makers of Orbit gum were very likely not inspired by maps in dynamical systems.

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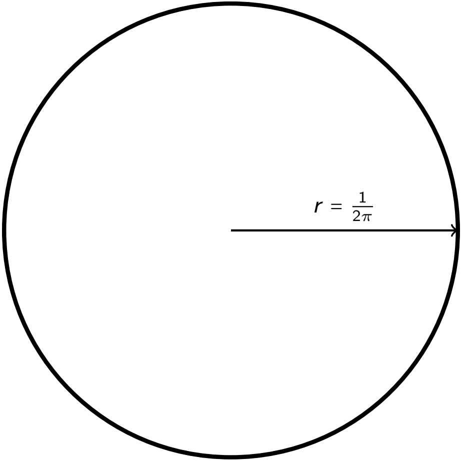

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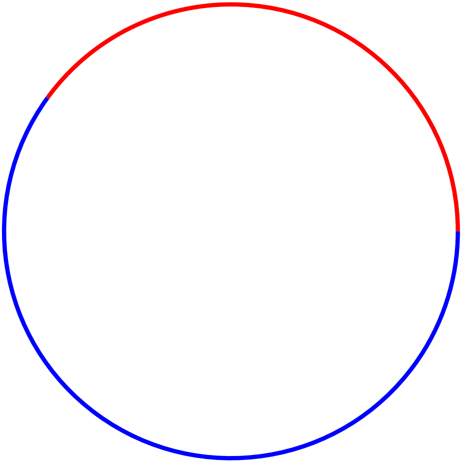

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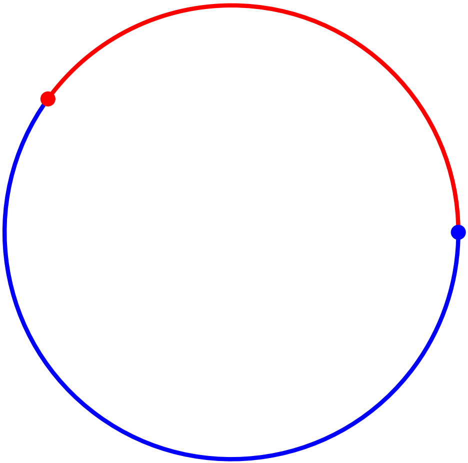

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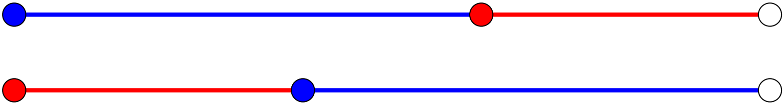

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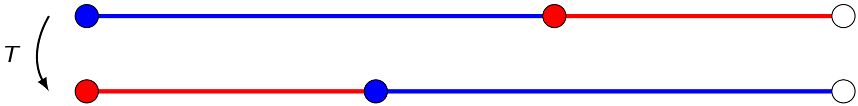

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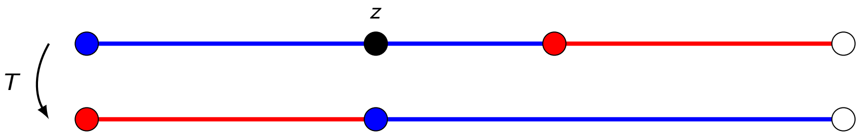

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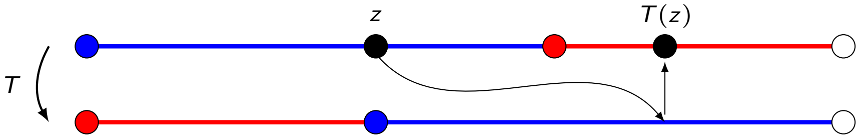

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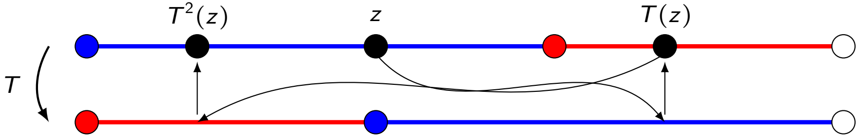

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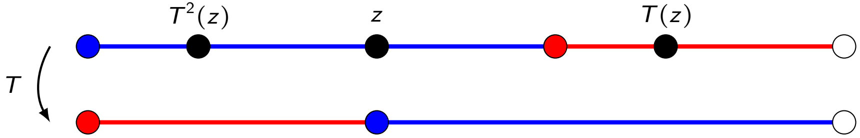

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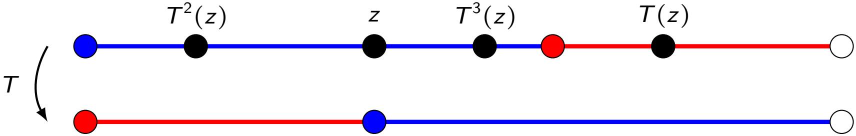

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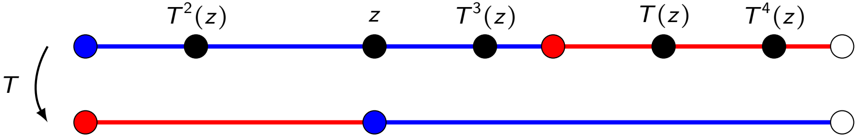

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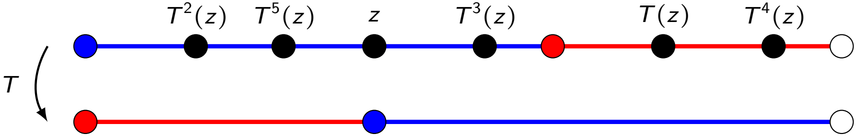

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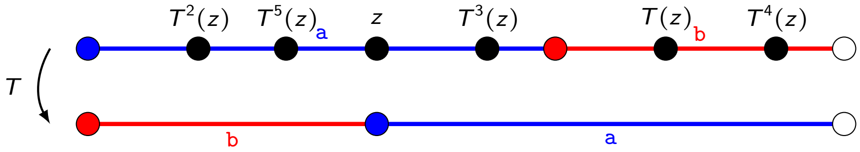

Interval Exchange Transformations

Definition (informal)

An interval exchange transformation (IET) is a map $T : [a, b) \to [a, b)$ defined by $T(z) = z + \tau_a$ if $z \in I_a$, where τ_a is the horizontal translation for all $z \in I_a$ such that subintervals are swapped (i.e. our mapping remains bijective and subintervals remain contiguous).

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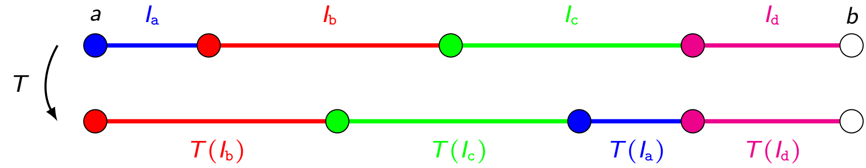

Note : it is natural to then associate a permutation π of the interval labels to an IET.

What Does It Mean For an IET To Be "Nice?"

We will define two terms.

Definition

Let $T : [a, b) \to [a, b)$ be an interval exchange transformation. We say that T is regular if, for every letter-labeled subinterval $[x_i, x_{i+1})$ in $[a, b)$ with $x_i \neq a$, $\bigcap_i \mathcal{O}(x_i) = \varnothing$. The latter equality is known as the "infinite disjoint orbit condition," or i.d.o.c. for short.

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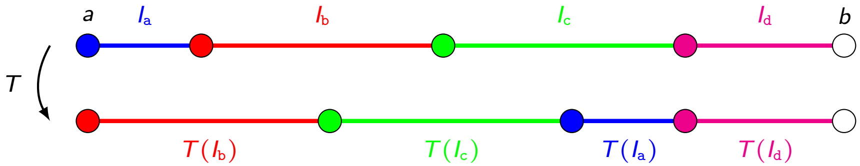

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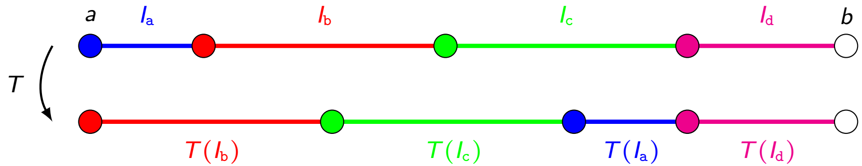

No ! The rightmost intervals on the top and bottom overlap, and their left endpoints share the same orbit !

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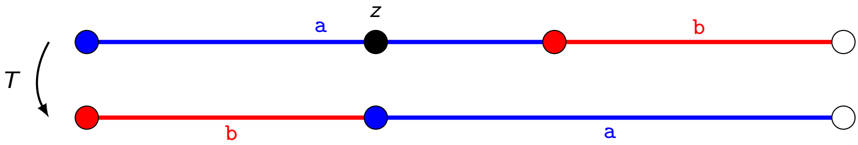

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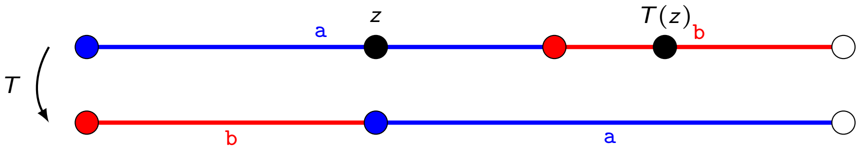

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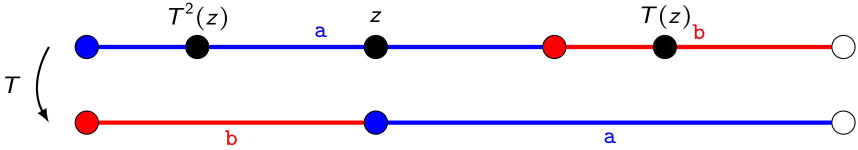

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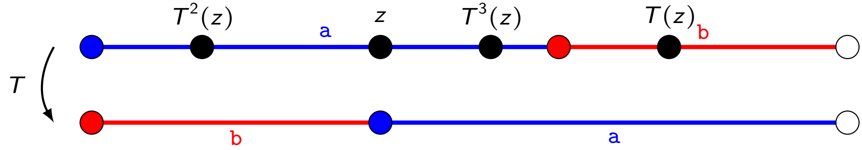

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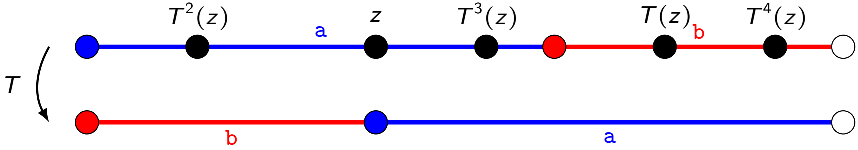

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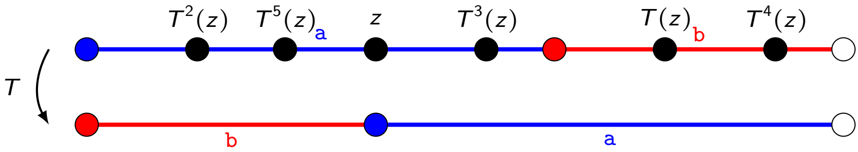

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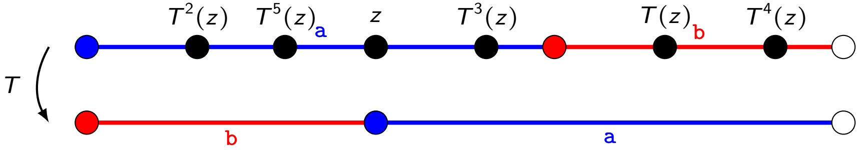

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The Language of an IET

Definition (informal)

Let $T : [a, b) \to [a, b)$ be an interval exchange transformation. Let $x \in [a, b)$. If T is minimal, we define the language $\mathcal{L}(T)$ of T to be the set of factors of $O_T(x)$. If T is not minimal, we define its language $\mathcal{L}(T)$ to be the union over all $x \in [a, b)$ of finite contiguous blocks of $O_T(x)$.

Example

In the previous IET, we had $\mathcal{O}_T(z) = \mathtt{abaaba}\ldots$. We see that $\mathtt{aab}$ is in the language of this IET, but $\mathtt{aaaaa}$ and $\mathtt{abaaba}\ldots$ are not.

Return Words

Definition (informal)

Let $T : [a, b) \to [a, b)$ be an interval exchange transformation. Let $x \in [a, b)$. Consider the language $\mathcal{L}(T)$ of T. Let $w \in \mathcal{L}(T)$. A (right) return word u to w in $\mathcal{L}(T)$ is a word such that $wu \in \mathcal{L}(T)$ has exactly two occurrences of w (at the beginning and at the end).

Example

In the language $\mathtt{abaaba}\ldots$, we have that $\mathtt{aab}$ is a return word to $\mathtt{ab}$.

The Return of the Burrows-Wheeler Transform

The Burrows-Wheeler transform gives us a nice way to characterize how "nice" certain factors in a language are.

Question

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Staring Down the IET Abyss

The world of IETs is much larger than the island we have loosely charted today. For example, all of the following are currently being studied from multiple mathematical perspectives :

- Equivalent IETs on subintervals of $[a, b)$

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- and much more !

Thank you for your attention !