

Induced Dynamics of Interval Exchange Transformations

Christian B. HUGHES

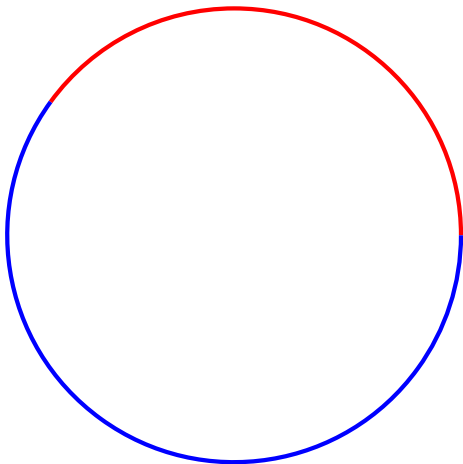


Czech Technical University in Prague

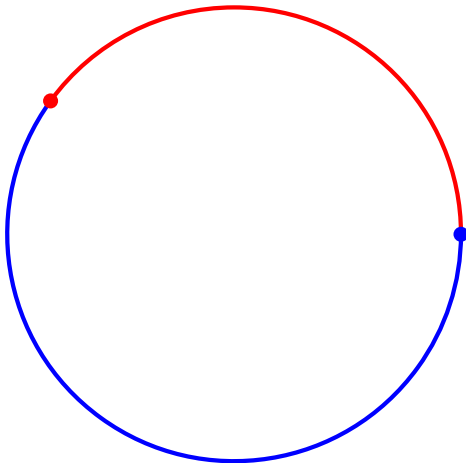
Janov nad Nisou, May 2026



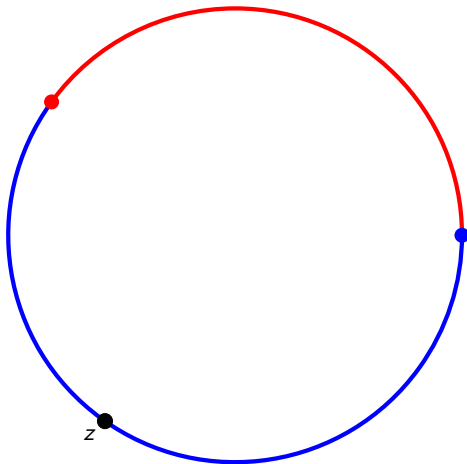
Rotation on a Circle



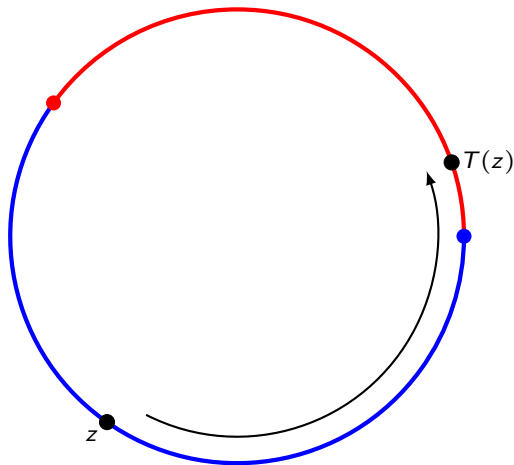
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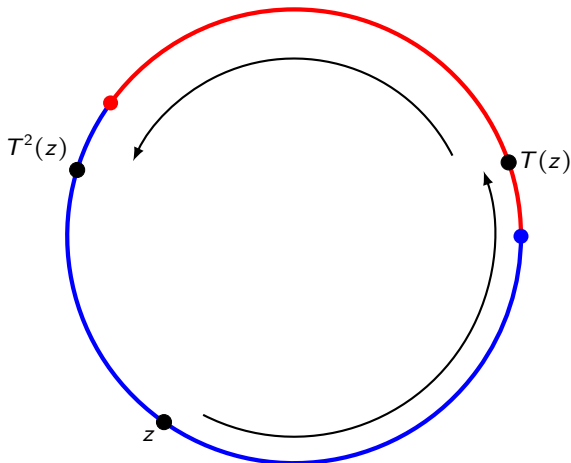
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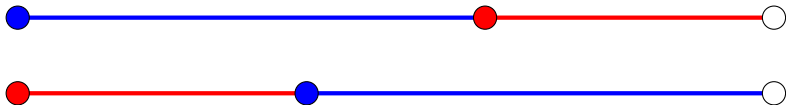
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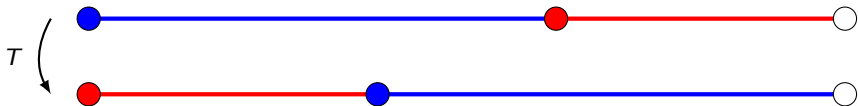
From Rotation to Interval Exchanges



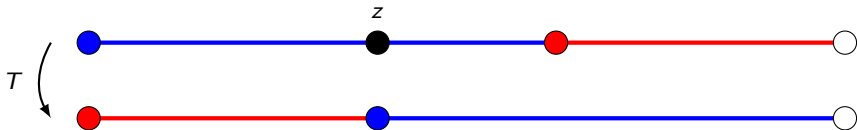
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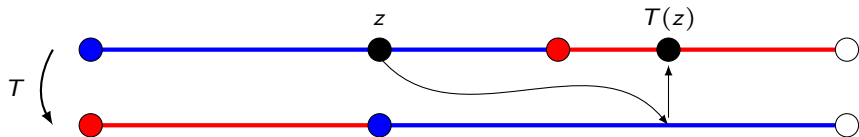
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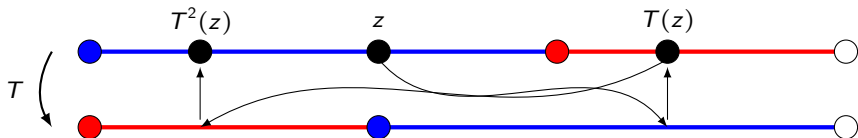
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Algebraically Describing the Example

Suppose the previous circle has circumference 1. We define

$$T: [0, 1) \rightarrow [0, 1), \quad T(z) = (z + \alpha) \bmod 1,$$

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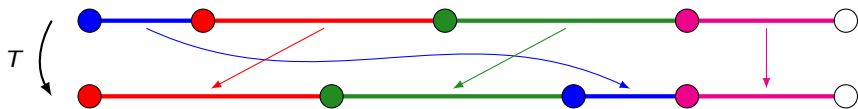
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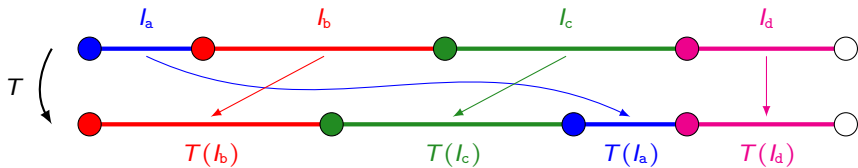
This can be viewed as an *exchange of two intervals*.

Question : Can we exchange *more than two* intervals?

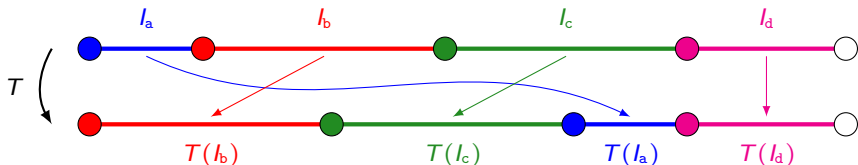
An Exchange of Four Intervals



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An Exchange of Four Intervals



Note that the permutation on the letters is $\pi = \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}$.

Interval Exchange Transformations

The previous examples lead us to the following definition.

Definition (interval exchange transformation)

Let $I = [\ell, r)$ be a real interval, with $\ell < r$. Let $\mathcal{A} = \{a_1 < \dots < a_n\}$ be an alphabet, and let $\mathcal{P} = \{I_{a_1}, I_{a_2}, \dots, I_{a_n}\}$ be a partition of I into $n \geq 1$ intervals of the form $I_{a_i} = [z_i, z_{i+1})$, $\ell \leq z_i < z_{i+1} \leq r$, $0 \leq i \leq n-1$.

Suppose $\pi \in S_{\mathcal{A}}$. The map $T : [\ell, r) \rightarrow [\ell, r)$ defined by $T(z) = z + \tau_a$, $z \in I_a$, where $\tau_a = \sum_{\pi^{-1}(b) < \pi^{-1}(a)} |I_b| - \sum_{b < a} |I_b|$, is called an n -interval exchange transformation.

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We often abbreviate " n -interval exchange transformation" by " n -IET."



Describing IET Behavior

Given an n -IET on an interval I , points of I can traverse the sub-intervals of I differently under T . Thus, we would like to describe how individual points $z \in I$ behave under T .

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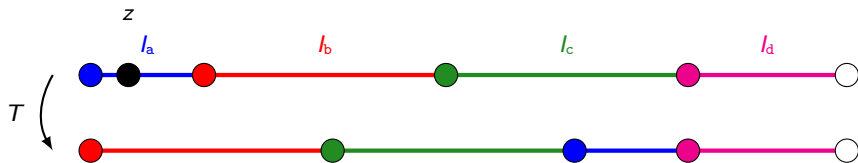
Orbits do not see the order of interval traversal of a point under T . The following related notion does.

Definition (trajectory of a point)

Let T be an IET on I , where I is partitioned into the sub-intervals $I_{a_1}, \dots, I_{a_n} \subset I$. The trajectory $\Omega_T(z)$ of a point $z \in I$ under T is the (right-infinite) word $\Omega_T(z) = b_0 b_1 \dots b_j \dots$, where $b_j = a$ if $T^j(z) \in I_a$.

More on Trajectories

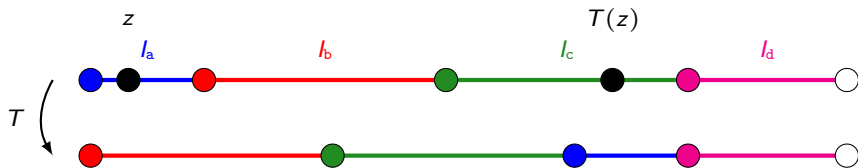
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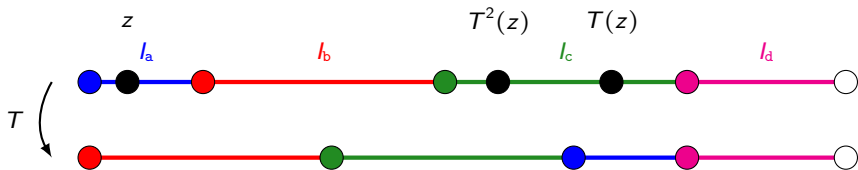
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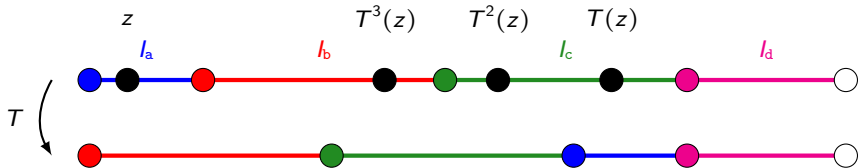
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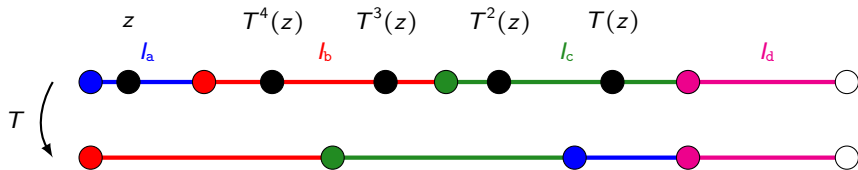
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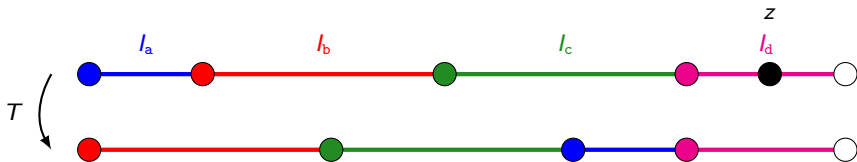
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Trajectories May Vary

Now let us instead trace a trajectory starting from a point $z \in I_d$.

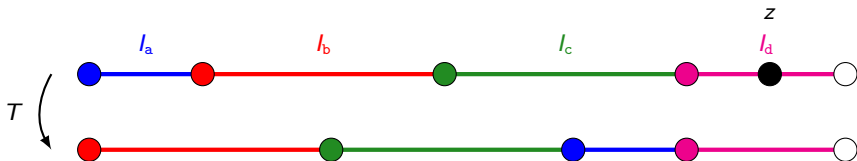


$$\Omega_T(z) = d$$

Trajectories May Vary

Now let us instead trace a trajectory starting from a point $z \in I_d$.

$$T^k(z) = z, \forall k \in \mathbb{Z}$$



$$\Omega_T(z) = \text{dd} \dots$$

Minimal Interval Exchanges

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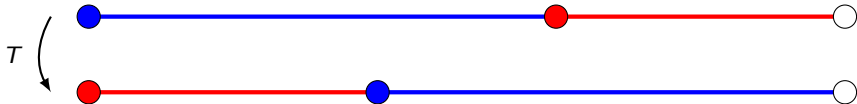
Definition (regular interval exchange)

An IET T is said to be *regular* if the orbits of all formal discontinuity points are infinite and pairwise disjoint.

A regular IET is always minimal, but the converse is not true.

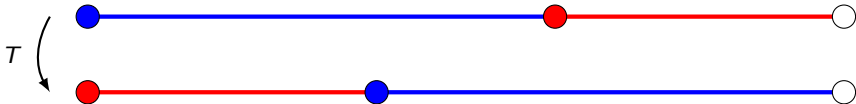


Interval Exchange Guessing Game



Question : Regular, minimal, or neither ?

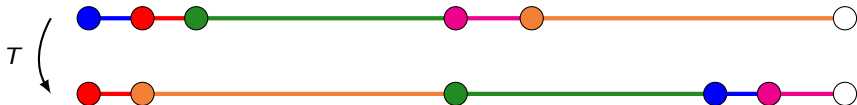
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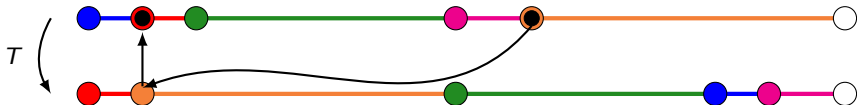
Answer : Regular (and thus also minimal).

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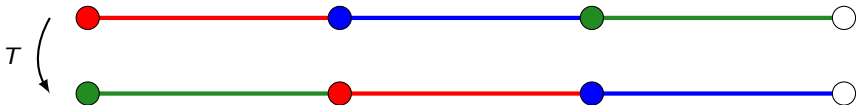
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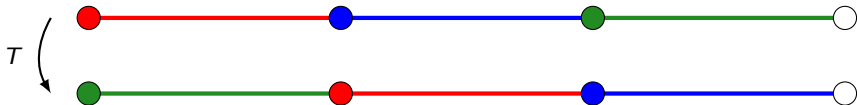
Answer : Minimal, but not regular.

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Answer : Neither.

Languages of Interval Exchanges

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Definition (language of an IET)

The language $\mathcal{L}(T)$ of the IET $T : I \rightarrow I$ is the set

$$\mathcal{L}(T) = \bigcup_{z \in I} \mathcal{L}(\Omega_T(z)).$$

We will return to $\mathcal{L}(T)$ when discussing return words and induction.

Intervals Associated to Factors

Given an IET T on I and a word $w \in \mathcal{L}(T)$, we can associate a sub-interval $I_w \subset I$ to w .



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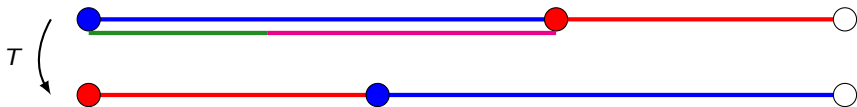
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Definition (cylinder)

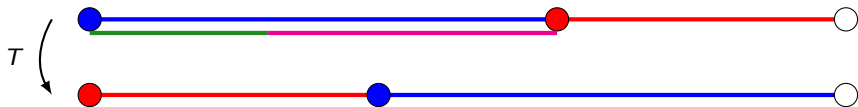
Given an IET T on I and a word $w = w_0 w_1 \dots w_{n-1} \in \mathcal{L}(T)$, the *cylinder* associated to w is the interval

$$I_w = I_{w_0} \cap T^{-1}(I_{w_1}) \cap \dots \cap T^{-(n-1)}(I_{w_{n-1}}).$$

First-Return Maps

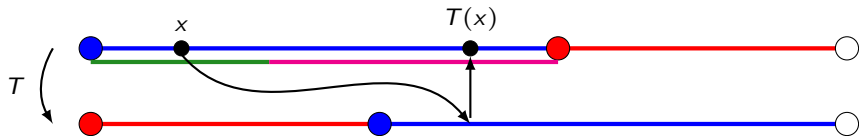


First-Return Maps



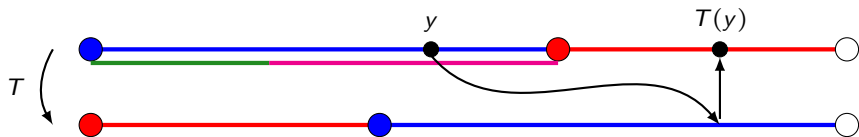
Question : How long do points in the **green** and **magenta** parts take to return to the **blue** interval ?

First-Return Maps



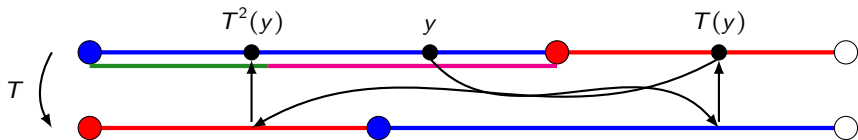
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Answer : Points in the **green** part return in one step, while points in the **magenta** part return in two steps.

First-Return Maps

Motivated by the previous example, we note that the return behavior of interval exchanges partitions intervals into constituent sub-intervals.

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Definition (first-return time)

Let T be an IET on I with $I' \subset I$ a sub-interval of I , and let $z \in I'$. The *first return time* of z to I' under T is the positive integer

$$\nu_{I', T}(z) = \min\{n > 0 \mid T^n(z) \in I'\}.$$

The map $\nu_{I', T} : I' \rightarrow \mathbb{Z}^+$ is called the first-return time map of T to I' .

Induced Maps

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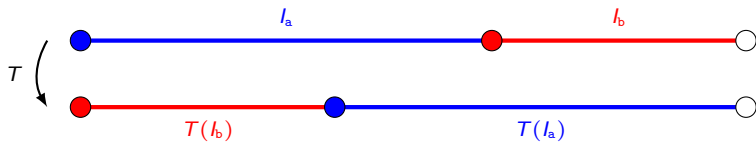
Definition (induced interval exchange)

Let T be an IET on I , with $I' \subset I$ a sub-interval of I . The transformation *induced* by T on I' is the map $T' : I' \rightarrow I'$ defined by

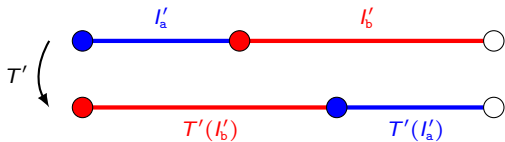
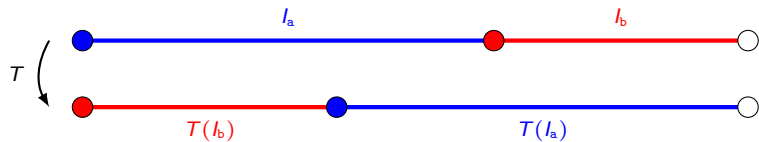
$$T'(z) = T^{\nu_{I', T}(z)}(z)$$

for points $z \in I'$.

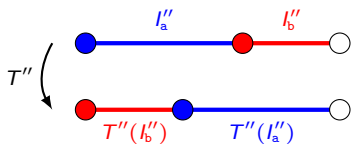
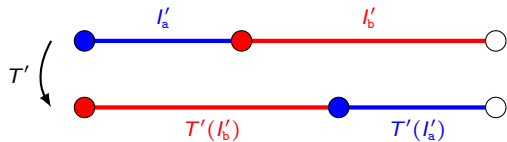
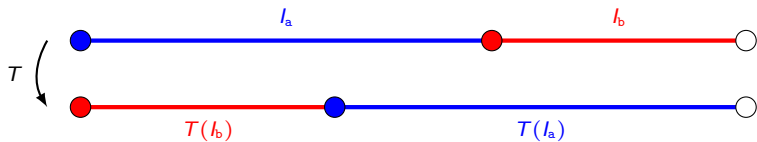
Induced Rotations



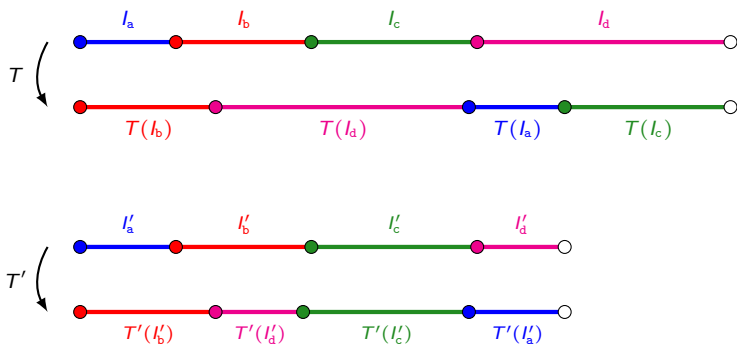
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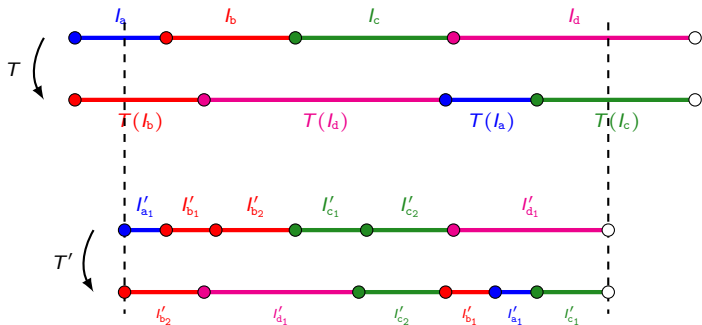


A Larger Example



In this example, we induce on the interval $I_a \cup I_b \cup I_c \cup I'_d$.

Induction is Not Always Nice



(Right) Rauzy Induction

Definition (Rauzy induction)

Let T be a regular n -IET α and β be the rightmost letters in the top and bottom rows. If $|I_\alpha| > |I_\beta|$, then the *Rauzy induction* of T is the first-return map of T to

$$I' = [\ell, r - |I_\beta|).$$

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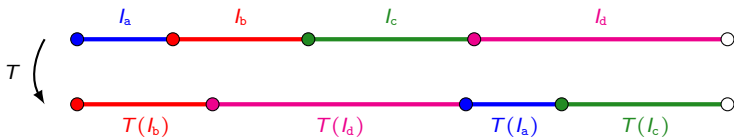
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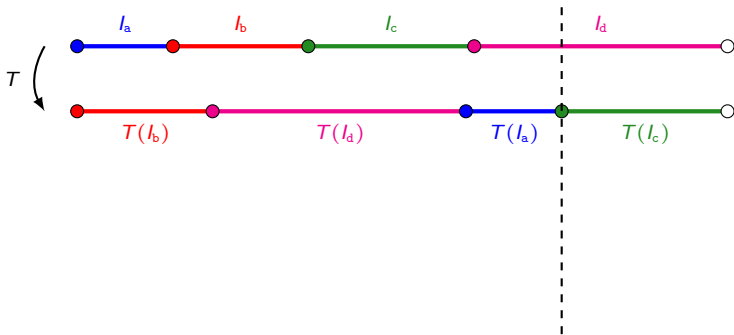
$$I' = [\ell, r - |I_\alpha|).$$

In other words, we compare the two rightmost exchanged intervals and remove the shorter one from the right.

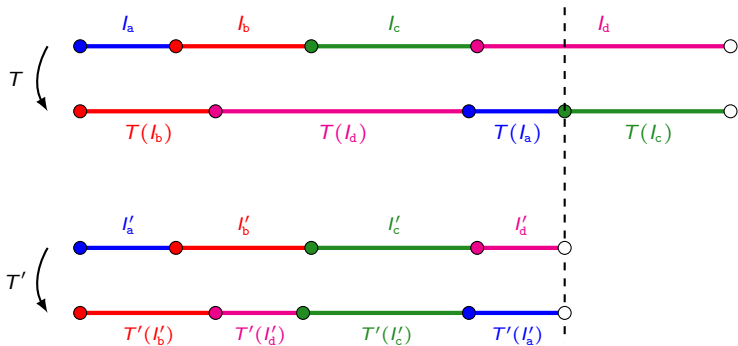
A Rauzy Induction Example



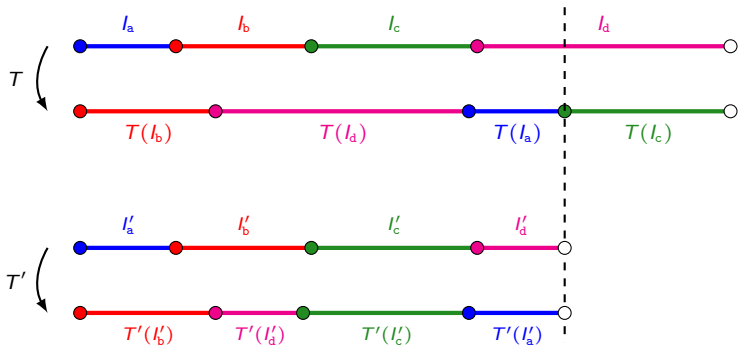
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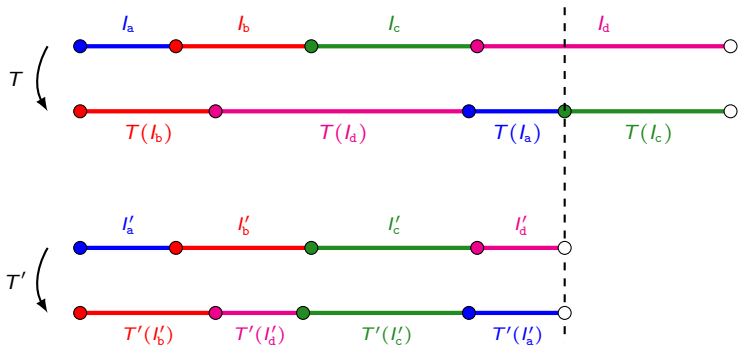
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Since $|I'_d| > |I'_c|$, d wins and c loses. The Rauzy-induced map is again a regular 4-IET. In fact, this holds for any regular n -IET.

Branching Rauzy Induction

Why only cut from one side?

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Definition (Branching Rauzy induction - informal)

Let T be a regular n -IET on I . We call the first-return map obtained by removing the shorter of the two *rightmost* intervals from the right end a *right Rauzy induction*, and the first-return map obtained by removing the shorter of the two *leftmost* intervals from the left end a *left Rauzy induction*.

The *branching Rauzy induction* of T is the choice, at each step, of performing either a left or a right Rauzy induction.

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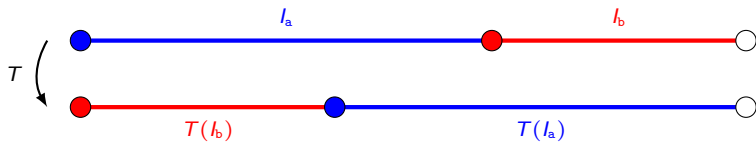
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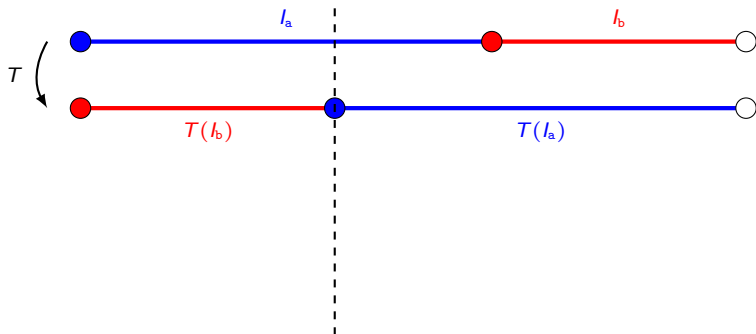
We call the right inductions ρ steps, and the left inductions λ steps.



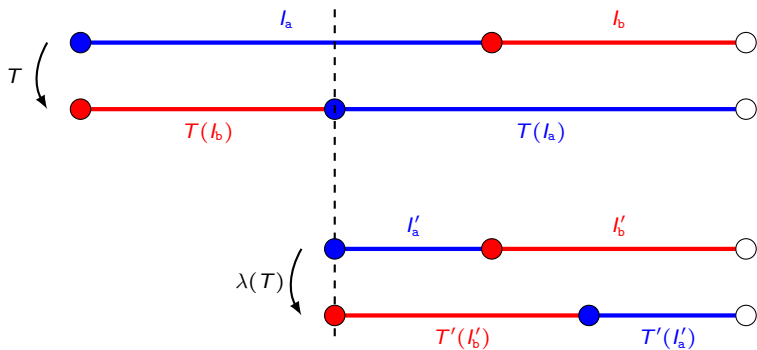
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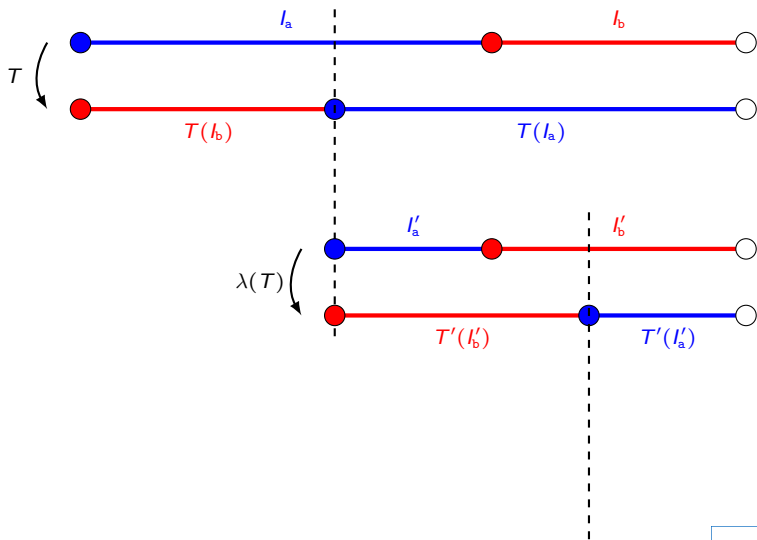
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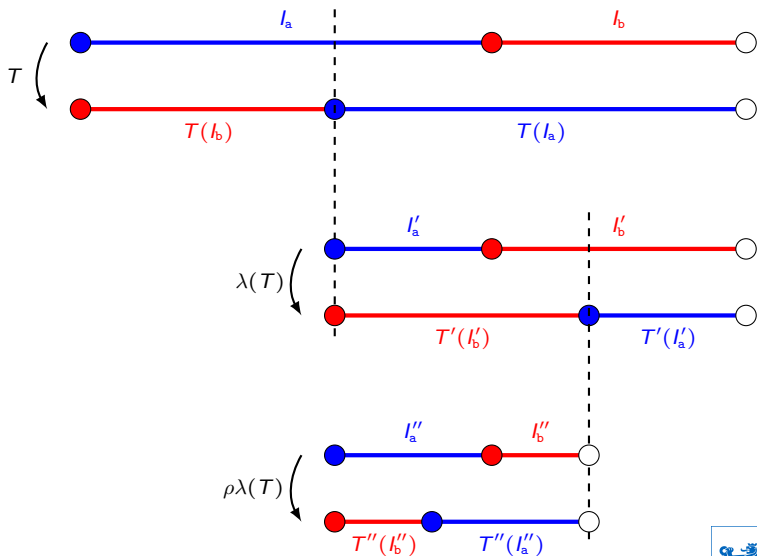
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Properties of Branching Rauzy Induction

Proposition (Dolce, Perrin 2018)

Let T be a regular n -IET on I . Let $w \in \mathcal{L}(T)$. There exists a finite sequence χ of ρ/λ steps such that $\chi(T)$ is the induced transformation of T on I_w .



Induced Interval Exchanges and Trajectories

Each induction step relates the trajectory of the induced map to the trajectory of the original.

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Proposition (relation of trajectories)

Let $T' = \chi(T)$ be obtained from T by a single Rauzy step $\chi \in \{\rho, \lambda\}$. Then there is a morphism φ_χ such that

$$\Omega_T(z) = \varphi_\chi(\Omega_{T'}(z)) \quad \text{for every } z \in I'.$$

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If a right step keeps **a** and absorbs **b** on its right, the associated morphism is

$$\alpha_{a,b}: a \mapsto ab, \quad c \mapsto c \quad (c \neq a).$$

We could also have the morphism $\tilde{\alpha}_{a,b}: a \mapsto ba$.

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Reading the induced trajectory and applying φ_χ recovers the original coding.

Induced Interval Exchanges and Return Words

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When T is regular, I_w is reached by some $\chi \in \{\rho, \lambda\}^*$. Composing the step morphisms φ_χ and applying the result to the induced alphabet produces precisely the return words to w .

Induction on Non-Regular Interval Exchanges

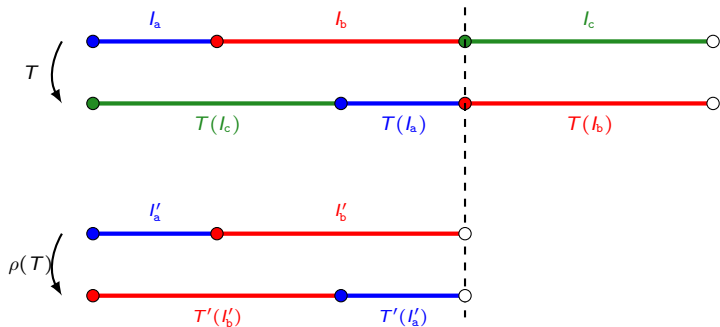
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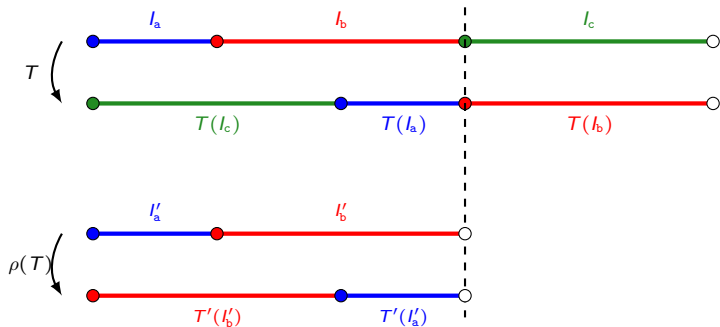
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Here c is absorbed into b . The corresponding morphism is $b \mapsto bc$.

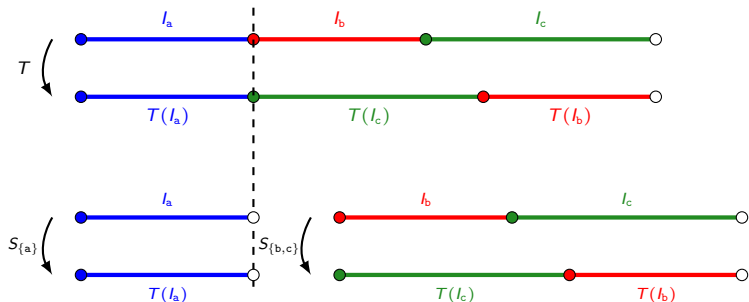
Induction on Non-Minimal Interval Exchanges

A reducible permutation has an invariant block that Rauzy and merging steps cannot reach. We isolate it with a *splitting* step.



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Induction on Cylinders of IETs

Proposition (Dolce, H. 2026+)

Let T be an interval exchange transformation. Let $w \in \mathcal{L}(T)$. There exists a finite sequence of extended branching Rauzy induction steps χ such that $\chi(T)$ is the induced map of T on I_w .

Thank you for your attention

