Bases and Dimension

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Contents

1 More on Span and Linear Independence

1.1 A Brief Review of Span

Previously, we explored span and linear independence. Although these notions seemed roughly motivated by similar ideas at that time, it turns out that they are more closely related than they appeared.

Recall that the span of a collection of vectors in a vector space V can be thought of as describing the unique linear/planar/hyper-planar subspace in V containing that collection of vectors.

Example 1.1. *Consider the vectors* $\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} \in \mathbb{R}^3$ *and* $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^3$. *We have*

$$span\left\{\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}\right\} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix},\ c_1, c_2 \in \mathbb{R}\right\}.$$

Our chosen vectors in $\mathbb{R}^3$ *"point" in different "directions" (i.e, they are linearly independent), and thus this spanning set forms a plane in* $\mathbb{R}^3$*:*

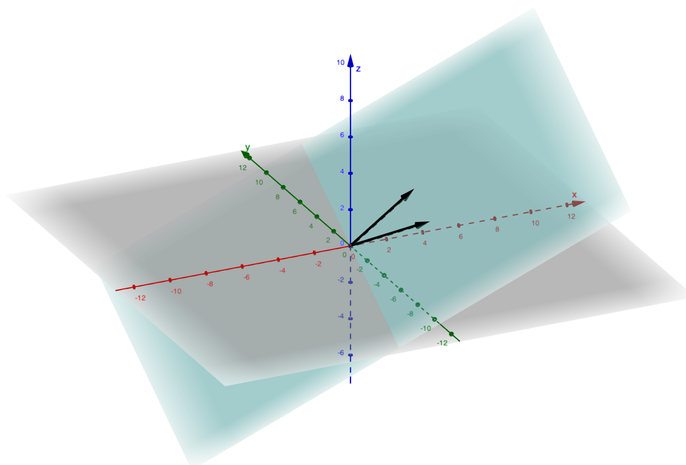

Figure 1: The plane spanned by $\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} \in \mathbb{R}^3$ and $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^3$.

Remark 1.1. *As always, we do not recommend relying on visual-spatial reasoning in general to determine the structure of the spanning set of a given collection of vectors. Here, it simply serves as a valuable tool for building intuition.*

More formally:

Definition 1.1 (Span)**.** *Let V be a vector space over the field $\mathbb{F}$. Let $\{v_1, \dots, v_n\} \subset V$. We define*

$$\mathit{span}\{v_1, \dots, v_n\} = \{v \in V : v = c_1 v_1 + \cdots + c_n v_n,\ c_1, \dots, c_n \in \mathbb{F}\}.$$

If $\mathit{span}\{v_1, \dots, v_n\} = V$, we say that the collection $\{v_1, \dots, v_n\}$ spans *V.*

But what can happen when we choose many vectors (and thus many "directions" in our vector space V)? We know in $\mathbb{R}^3$, for example, that choosing one vector and taking its span gives us a line (if we do not choose the empty list or the zero vector). Choosing two vectors and taking their span gives us a plane if the vectors are linearly independent. What about three vectors? Barring linearly dependent cases, all linear combinations (the span) of the vectors in that list would simply yield all of $\mathbb{R}^3$. (Intuitively, $\mathbb{R}^3$ has three axes, and our three vectors pointing in three separate directions function as "axes" of our own, with which we can reach any other vector in $\mathbb{R}^3$.)

At this point, adding more vectors to this spanning collection would be redundant in the sense that such a vector is already in the span of (i.e, "reachable" by) the previous vectors in collection. Thus, the notion of a "minimal spanning list" naturally interests us - more on that later.

1.2 More on Span

Inspired by the observation concerning the redundancy of the inclusion of additional vectors in a spanning list, we have the following proposition:

Proposition 1.1. *Let V be a vector space. Let $S = \{v_1, \dots, v_n\} \subset V$. There exists a set $S' \subset S$ such that the vectors in S' are linearly independent and spanS' = span$S$.*

The proof of this proposition was given in the solution set to the exercises on span and linear independence. We repeat it here for the sake of completeness.

Proof. Let $S = \{v_1, \dots v_n\}$ span V. If S is already linearly independent, then set $L' = L$, and we are done. If S is not linearly independent (i.e., linearly dependent), proceed as follows:

Begin with $S_0 = \varnothing$ (the empty list). For $i = 1, \dots, n$, if $v_i \notin \text{span}(S_{i-1})$, set $S_i = S_{i-1} \cup \{v_i\}$. Otherwise, set $S_i = S_{i-1}$ (that is, drop v_i from the list). Let $S' = S_n$. Because the list is finite, this process ends.

Evidently, $S' \subset S$, so span$S' \subset$ span$L$. Conversely, every dropped v_i lies in spanL' by definition. Every dropped v_i was in span$S_{i-1} \subset$ spanS' at the moment it was dropped. Hence, each $v_i \in$ spanS', so span$S \subset$ spanS', so spanS' = span$S$.

The list L' is linearly independent by construction, since whenever we add a new vector, it is not in the span of the vectors already kept. This is exactly the criterion for linear independence.

Thus, S' is a linearly independent list with spanS' = span$S$, and we are done. □

Intuitively, this proposition tells us that any spanning set is either "redundant," in the sense that it contains vectors which are already in the span of the others in the set, or just barely non-redundant, in the sense that set is already linearly independent, yet still spans the entire space.

Until now, we have discussed span properties in terms of given lists of vectors (i.e., we have asked about the properties of the space given by the span of a set of vectors, or we have asked when a spanning list can be reduced to a linearly independent list). However, in mathematics, we often ask if we can directly relate such properties to the object of central interest; in linear algebra, we are primarily interested in vector spaces[1] and linear maps between them[2].

Given a vector space V, can we say something about the span of any subset of vectors of V? First, recall that in this course, we restrict ourselves to spaces "like" $\mathbb{F}^n$, so we will consider $\mathbb{F}^n$ in lieu of a general vector space V.

Proposition 1.2. *Let $\{v_1, \dots, v_m\} \subset \mathbb{F}^n$. Suppose that span$\{v_1, \dots, v_m\} = \mathbb{F}^n$. Then $m \geq n$.*

[1]Technically, we are only interested in *finite-dimensional* vector spaces here. See the "dimension" section for more details.

[2]This idea holds in general: vector spaces and linear maps, sets and functions, spaces with "distance" and continuous functions, and so on. In mathematics, we try to tie properties to the objects of interest or to the "natural" maps between them.

In other words, Proposition 1.2 asserts that a set of vectors which, when combined in every possible way, covers all n "directions" of $\mathbb{F}^n$, must possess at least n vectors. For example, consider any two vectors in $\mathbb{R}^3$. The span of these vectors will at most form a plane.

Remark 1.2. *Note that Proposition 1.2 is an implication, not an equivalence. It gives us a necessary condition for a set of vectors in $\mathbb{F}^n$ to span $\mathbb{F}^n$, but not a sufficient condition. For example, the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$ does not span $\mathbb{R}^3$, despite containing three distinct vectors in $\mathbb{R}^3$.*

However, this proposition does *give us a way to rule out whether a set of vectors in $\mathbb{F}^n$ spans $\mathbb{F}^n$: if the given set contains less than n vectors, then it cannot span $\mathbb{F}^n$.*

1.3 A Brief Review of Linear Independence

The notion of whether a list[3] $(v_1, \ldots, v_n)$ of vectors has redundancies is captured by the notion of linear (in)dependence.

Example 1.2. *Consider the vectors $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \in \mathbb{R}^3$ and $\begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} \in \mathbb{R}^3$. These vectors are clearly scalar multiples of each other ($2 \cdot \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix}$), and thus they "point" in the same "direction" in $\mathbb{R}^3$.*

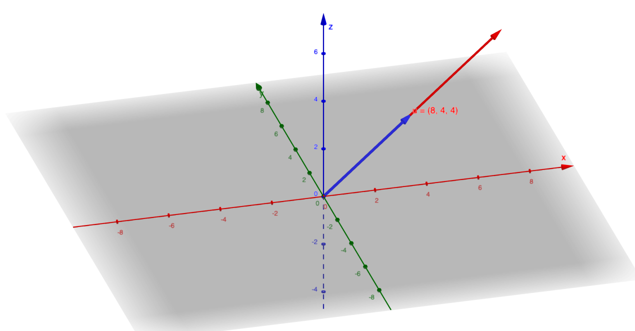

Figure 2: The vectors $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ (blue arrow) and $\begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix}$ (red arrow) in $\mathbb{R}^3$.

[3]One might notice that we have made the distinction between sets and lists of vectors throughout these notes. We make this distinction for two reasons. Firstly, we would like for a vector to be linearly dependent with itself, and consequently we must have some structure which allows for multiple copies of identical elements. Secondly, many processes through which we obtain one list from another require a notion of order (i.e., the "first" vector and the "second" vector in the collection).

Since $2 \cdot \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix}$, *we can write*

$$2 \cdot \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In particular, the fact that the vectors are scalar multiples of each other implies the existence of a linear combination of the vectors summing to zero, with some coefficients nonzero. Directly determining whether many vectors are scalar multiples of one another proves time-consuming for large lists of vectors, so we formalize this idea algebraically.

Definition 1.2 (Linear independence)**.** *Let V be a vector space over the field $\mathbb{F}$. A list of vectors $(v_1, \ldots, v_n)$ is said to be linearly independent if the only choice of $c_1, \ldots, c_n \in \mathbb{F}$ such that*

$$c_1 v_1 + \cdots + c_n v_n = 0$$

is $c_1 = \cdots = c_n = 0$. A list of vectors which is not linearly independent is said to be linearly dependent.

1.4 More on Linear Independence

In the previous section, we obtained a linearly independent list from a spanning list. We generalize Proposition 1.1 to any list of linearly dependent vectors.

Proposition 1.3 (Linear dependence lemma)**.** *Let V be a vector space over the field $\mathbb{F}$. Let $(v_1, \ldots, v_n)$ be a linearly dependent list of vectors in V with $v_1 \neq 0$. There exists $j \in \{2, \ldots, n\}$ such that:*

1. $v_j \in \mathit{span}(v_1, \ldots v_{j-1})$.
2. $\mathit{span}(v_1, \ldots v_{j-1}, v_{j+1}, \ldots, v_n) = \mathit{span}(v_1, \ldots, v_n)$.

Although the proof of Proposition 1.3 is relegated to Linear Algebra 2, we recommend attempting the proof as an exercise.

Similarly to Proposition 1.1, this proposition tells us that pruning redundant vectors (i.e., vectors which are linear combinations of other vectors in the list) from a list does not affect the spanning properties of the list. Intuitively, this should feel natural; the subspace given by span is, in some sense, a measure of how many directions one has to work with. For example, if one has two linearly independent vectors in $\mathbb{R}^3$, then one has two distinct "directions" (i.e., the two vectors are not scalar multiples of each other). If I add multiple copies of one "direction," then I have redundancies; one is enough!

Remark 1.3. *Note that Proposition 1.3 is actually an equivalence. That is, having a linear dependent list implies we can remove vectors from the list and retain the same span, but being able to remove vectors from the list and retain the same span also implies that our original list was linearly dependent.*

Example 1.3. *Consider the list of vectors* $([\begin{smallmatrix}3\\-2\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}], [\begin{smallmatrix}2\\1\end{smallmatrix}])$ *in* $\mathbb{R}^2$*. How can we obtain a span-preserving linear independent list from this list of vectors? First, note that this list of vectors is linearly dependent. Indeed,*

$$\begin{bmatrix}2\\1\end{bmatrix} = 2/3 \cdot \begin{bmatrix}3\\-2\end{bmatrix} + 7/3 \cdot \begin{bmatrix}0\\1\end{bmatrix}.$$

By part 2 *of Proposition 1.3 with* $j = 3$*, we have*

$$span\left(\begin{bmatrix}3\\-2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = span\left(\begin{bmatrix}3\\-2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right).$$

But is the list $([\begin{smallmatrix}3\\-2\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}])$ *linearly independent?*

Before applying Gaussian elimination, we should think about these vectors. Our goal is to find a counterexample (i.e., a choice of not identically zero scalars such that the associated linear combination of these vectors is zero). For the first slot to be zero, note that the first vector in the modified list must have its first entry equal to zero, and this forces $c_1 = 0$*, which also forces* $c_2 = 0$*, since the second entry of the second vector must also be zero.*

Consequently, the list $([\begin{smallmatrix}3\\-2\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}])$ *is linearly independent, and we are done.*

In the case of Example 1.3, we were able to reason about the vectors to obtain a linearly independent list. In general, however, this is not sufficient. How can we find such lists systematically? Gaussian elimination proves useful.

Let $(v_1, \ldots, v_n)$ be a list of vectors in the vector space V. We create the matrix

$$M = [v_1, \ldots, v_n]$$

which is simply the matrix having as columns the vectors in $(v_1, \ldots, v_n)$. We apply Gaussian elimination to obtain the matrix M in row-echelon form. If, following this reduction, there is some variable that can be rewritten in terms of the others, we remove that variable (i.e., remove the vector in the corresponding column). We continue removing such variables until no additional variables can be removed.

Example 1.4. *Consider the list of vectors* $\left([\begin{smallmatrix}1\\1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\\1\end{smallmatrix}], [\begin{smallmatrix}0\\1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\1\\1\end{smallmatrix}]\right)$ *in* $\mathbb{R}^3$ *and form the matrix with these as columns:*

$$M = [v_1 \ v_2 \ v_3 \ v_4] = \begin{pmatrix}1 & 1 & 0 & 1\\1 & 0 & 1 & 1\\0 & 1 & 1 & 1\end{pmatrix}.$$

We apply Gaussian elimination to M*:*

$$R_2 \leftarrow R_2 - R_1 : \quad \begin{pmatrix}1 & 1 & 0 & 1\\0 & -1 & 1 & 0\\0 & 1 & 1 & 1\end{pmatrix}$$

$$R_3 \leftarrow R_3 + R_2 : \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

This is in row-echelon form. Let us change our point of view for a moment. If we were to solve

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = v_4,$$

we could form the matrix with v_1, v_2, v_3 *as columns and* v_4 *as the augmented right-hand side:*

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right).$$

Applying Gaussian elimination to this augmented matrix, we would ultimately obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array} \right),$$

Notice that this is the same row-echelon form we obtained for the matrix $M = [v_1\ v_2\ v_3\ v_4]$*, except now we have an additional column corresponding to* v_4*. The first three columns are pivot columns, and the fourth is not. This tells us that* v_4 *can be written as a linear combination of* $v_1, v_2,$ *and* v_3*, since if we continued the elimination process to reduced row-echelon form, we would find*

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array} \right),$$

which corresponds to the equations

$$c_1 = \tfrac{1}{2}, c_2 = \tfrac{1}{2}, c_3 = \tfrac{1}{2},$$

and hence

$$v_4 = \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2 + \tfrac{1}{2} v_3.$$

In general, the pivot columns identify a span-preserving linearly independent list, and any non-pivot columns can be discarded without changing the span.

In Proposition 1.2, we tied a property concerning the length of spanning lists in a vector space to the vector space itself. We can say something similar to this proposition for linearly independent lists. Again, we restrict ourselves to spaces "like" $\mathbb{F}^n$.

Proposition 1.4. *Let* $(v_1, \ldots, v_m)$ *be a linearly independent list in* $\mathbb{F}^n$*. Then* $m \le n$*.*

Intuitively, this proposition states that there cannot exist a list of vectors in a vector space with more "directions" than the space allows[4]. For example, $\mathbb{R}^2$ has

[4] We formalize this intuition in the next section.

two "directions," and thus there cannot be a list of three linearly independent vectors in $\mathbb{R}^2$.

While Proposition 1.3 provides us with a way to obtain a linearly independent list from a spanning list, we would similarly like to obtain a spanning list from a linearly independent list. We can infer from the inequalities in Propositions 1.2 and 1.1 that such a process will involve extending the given linearly independent list.

Proposition 1.5. *Let $(v_1, \ldots, v_n)$ be a linearly independent list in a vector space[5] V. The list $(v_1, \ldots, v_n)$ can be extended to a linearly independent list $(v_1, \ldots, v_m)$, $m \geq n$, with span$(v_1, \ldots, v_m) = V$.*

We leave the proof of Proposition 1.5 as an exercise.

The way to do such extensions systematically is to mimic the deletion process for the reduction to linearly independent lists in reverse. Before, we began with a spanning list, constructed a matrix with these vectors as columns, and applied Gaussian elimination to remove non-pivot columns. Now, we begin with a linearly independent list, place its vectors as columns of a matrix, and again perform Gaussian elimination. If the resulting row-echelon form contains zero rows, this indicates that some directions in the space are still missing from the span of our current list. We then append vectors of the form $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ to "patch" these missing directions.

1.5 Exercises

1. Extend the list $\left(\begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 6 \\ -5 \end{bmatrix}\right)$ to a spanning list of $\mathbb{R}^4$.

2. Reduce the list $\left(\begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$ to a linearly independent list in $\mathbb{R}^3$.

3. Extend the list $\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right)$ to a spanning list of $\mathbb{Z}_5^3$.

4. Provide two different spanning lists of $\mathbb{Z}_7^2$ and prove they span $\mathbb{Z}_7^2$.

5. ($\star$) The vectors $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}\right)$ span a certain subspace of $\mathbb{R}^3$ (i.e., the trivial subspace, a line, a plane, or $\mathbb{R}^3$ itself). Determine the kind of subspace they span. Reduce them to a span-preserving linearly independent list (if the list is not linearly independent already).

Bonus. Show Proposition 1.5.

[5] The vector space V is assumed to be finite-dimensional in the in the next two sections unless explicitly stated otherwise.

2 Bases and Dimension

In Propositions 1.4 and 1.2, we observed that, in $\mathbb{F}^n$, any spanning list of vectors has at least n members, while any linearly independent list of vectors has at most n members. This suggests that, in $\mathbb{F}^n$, there is something special about lists of vectors of length n. Such lists are "almost" too large to be linearly independent, yet as small as they can be to span $\mathbb{F}^n$. It would be prescient to give lists of length n which "barely" span $\mathbb{F}^n$, but are also "barely" linearly independent in $\mathbb{F}^n$, a name.

Definition 2.1 (Basis)**.** *Let V be a vector space and let $(v_1, \ldots, v_n)$ be a list of vectors in V. The list $(v_1, \ldots, v_n)$ is said to be a* basis *for the space V if $(v_1, \ldots, v_n)$ is linearly independent and spans V.*

Example 2.1. *Consider the list $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ of vectors in $\mathbb{R}^3$. Clearly this list of vectors spans $\mathbb{R}^3$, since for any $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \in \mathbb{R}^3$, we have*

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = c_1 \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Furthermore, this list is linearly independent; as an augmented matrix, the list has a pivot in every column. Consequently, $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ is a basis of $\mathbb{R}^3$.

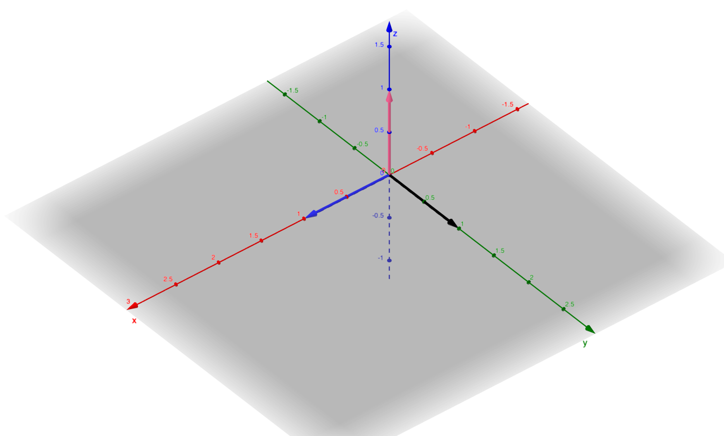

Figure 3: The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ (blue arrow), $\begin{bmatrix}0\\1\\0\end{bmatrix}$ (black arrow), and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ (pink arrow).

Note that the definition of basis immediately implies that all bases for a vector space have the same number of elements. Notice we said "all bases." Bases are certainly not unique. Indeed, multiplying each vector in the list of Example 2.1 by -1 (and, in fact, any non-zero scalar) also yields a basis.

The bases of a vector space form a sort of fundamental object for understanding the space. Bases are so fundamental, in fact, that their consistent length leads us to the following definition.

Definition 2.2 (Dimension of a vector space)**.** *Let V be a vector space. Let $(v_1, \ldots, v_n)$ be a basis of V. We say the vector space V has dimension n.*

The notion of dimension formalizes our intuition of "number of directions in a space." The space $\mathbb{R}^3$, for example, has three "directions," and thus, perhaps unsurprisingly, has dimension 3, $\mathbb{F}^n$ has dimension n, planes in $\mathbb{R}^3$ have dimension 2, and so on.

But why should this be the case? Why are these ideas of being both spanning and linearly independent, when coupled, yielding a notion of dimension so compatible with our intuition? Consider $\mathbb{R}^n$, and think of the largest spanning set of vectors you can think of. The largest such spanning set is just $\mathbb{R}^n$ itself. Similarly, think of the smallest linearly independent list: the empty list. What meaningfully distinguishes $\mathbb{R}^n$ from $\mathbb{R}^n \setminus \left\{ \left[\begin{smallmatrix} 104 \\ -35 \\ 82 \end{smallmatrix} \right] \right\}$? Similarly, what meaningfully distinguishes $\varnothing$ from $\varnothing \cup \left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix} \right]$? It appears that defining dimension in any other way would lead to some arbitrary choice of n. However, if we define dimension as in Definition 2.2, the dimension of the space is the only value for which we obtain a coincidence between spanning lists and linearly independent lists.

The importance of bases, as well as our intuition for them, are reaffirmed by the following two propositions, which tell us that "barely" spanning a vector space and "barely" being linearly independent are sufficient conditions for being a basis.

Proposition 2.1 (All minimal spanning lists are bases)**.** *Let V be a vector space of dimension n and let $(v_1, \ldots, v_n)$ be a spanning list for V. Then $(v_1, \ldots, v_n)$ is a basis for V.*

Proposition 2.2 (All maximal linearly independent lists are bases)**.** *Let V be a vector space of dimension n and let $(v_1, \ldots, v_n)$ be linearly independent list. Then $(v_1, \ldots, v_n)$ is a basis for V.*

These propositions inform us that bases are characterized both by their minimal spanning property and by their maximal linearly independent property.

Remark 2.1. *Etymologically, the word "basis" comes from the Latin noun "basis," which means a base or a foundation. Naturally, one can think of a basis for a vector space as the foundation from which all vectors in the space are generated.*

While we have defined basis and discussed some of its properties, we have not yet discussed how to computationally determine when a given list of vectors is a basis for a vector space. We determine when a list of vectors is a basis by verifying that the list is both spanning and linearly independent. Earlier, we saw how to verify these properties via Gaussian elimination, so the process for determining when a list of vectors is a basis is simple: form a matrix with the vectors of the list as its columns, apply Gaussian elimination to this matrix, and determine whether the list is linearly independent and spanning. If either of these conditions fail (i.e., the Gaussian elimination yields a zero row or a column with no pivots), the list of vectors is not a basis.

Example 2.2. *Consider the list of vectors* $\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$ *in* $\mathbb{R}^3$ *and form the matrix with these vectors as columns:*

$$M = [v_1 \ v_2 \ v_3] = \begin{pmatrix}1 & 0 & 1\\2 & 1 & 0\\3 & 2 & 1\end{pmatrix}.$$

We apply Gaussian elimination to M*:*

$$R_2 \leftarrow R_2 - 2R_1 : \quad \begin{pmatrix}1 & 0 & 1\\0 & 1 & -2\\3 & 2 & 1\end{pmatrix}$$

$$R_3 \leftarrow R_3 - 3R_1 : \quad \begin{pmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 2 & -2\end{pmatrix}$$

$$R_3 \leftarrow R_3 - 2R_2 : \quad \begin{pmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 2\end{pmatrix}.$$

This matrix has no zero rows, so the list spans $\mathbb{R}^3$*. Additionally, the matrix has a pivot in every column, so it is linearly independent. Consequently, the list* $\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$ *is a basis for* $\mathbb{R}^3$*.*

We conclude with a few facts concerning dimension. An important notion is that of finite dimensionality.

Definition 2.3 (Finite dimensional vector space)**.** *Let* V *be a vector space. The space* V *is said to be finite dimensional if there exists a finite list* $(v_1, \dots, v_n)$ *of vectors such that span*$(v_1, \dots, v_n) = V$*. Otherwise,* V *is said to be infinite dimensional.*

Definition 2.3 captures the intuitive idea that a space is finite in terms of "dimensional size" if one can reach every point in the space with only finitely many "directions." As an immediate consequence of this definition, we have the following proposition.

Proposition 2.3. *Let* V *be a finite dimensional vector space. All subspaces of* V *are also finite dimensional.*

Since a vector space can have infinitely many bases, it is probably a good idea to have some sort of "default" basis. With the notion of dimension in hand, we are prepared to define this formally.

Definition 2.4 (Standard basis)**.** *Let V be a vector space with dimension $n \in \mathbb{Z}^+$ over the field $\mathbb{F}$. The list of vectors $(v_1, \dots, v_n)$ with the property that v_i has a* 1 *in its i^{th} slot and* 0 *in all others is called the standard basis for V.*

2.1 Exercises

1. Is the list $\left(\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right)$ a basis for $\mathbb{R}^3$? Justify your response.

2. Extend the list $\left(\begin{bmatrix}2\\-7\\3\end{bmatrix}, \begin{bmatrix}5\\-3\\1\end{bmatrix}\right)$ to a basis for $\mathbb{R}^3$. Prove that the resulting list is a basis.

3. Provide a basis for $\mathbb{C}^2$ that is not a scalar multiple of the standard basis. Prove that it is a basis for $\mathbb{C}^2$.

4. What is the dimension of $M_{2,4}(\mathbb{R})$? Justify your response.

5. Modify the list $\left(\begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right)$ so that the resulting list is a basis for $\mathbb{Z}_7^2$. Prove that the resulting list is a basis.

6. Let $\mathbb{F}$ be a field. Provide a basis for $\mathbb{F}$. Prove that it is a basis for $\mathbb{F}$. Are there lists of elements in $\mathbb{F}$ which do not comprise a basis?

7. Extend the list $\left(\begin{bmatrix}1\\-4\\5\\2\end{bmatrix}\right)$ to a basis for $\mathbb{R}^4$. Prove that the resulting list is a basis.

8. Show that the same three vectors from Exercise 1 form a basis of $\mathbb{Z}_2^3$.

9. Provide a list of vectors that is a basis for $\mathbb{R}^2$, but not for $\mathbb{Z}_5^2$. Prove that it is a basis for $\mathbb{R}^2$, but not for $\mathbb{Z}_5^2$

10. ($\star$) A finite dimensional vector space cannot contain an infinite dimensional subspace, but can an infinite dimensional vector space contain a finite dimensional subspace?

11. ($\star$) Show that the list $\left(\begin{bmatrix}k-1\\k\\k\\k\end{bmatrix}, \begin{bmatrix}k\\k-1\\k\\k\end{bmatrix}, \begin{bmatrix}k\\k\\k-1\\k\end{bmatrix}, \begin{bmatrix}k\\k\\k\\k-1\end{bmatrix}\right)$, $k \in \mathbb{Z}$, $k \geq 1$, is a basis for $\mathbb{R}^4$.

12. True or false: there is a spanning list of length 2 in $\mathbb{R}^4$. Justify your response.

13. True or false: there is a linearly independent list of length 9 in $\mathbb{R}^{15}$. Justify your response.

Bonus. Deduce the probability that any two random vectors in $\mathbb{R}^2$ do not form a basis.