Solutions to Bases and Dimension

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1 Span and Linear Independence

1. Extend the list $\left(\begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 6 \\ -5 \end{bmatrix}\right)$ to a spanning list of $\mathbb{R}^4$.

Solution. We form a matrix with our two vectors $\left(\begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 6 \\ -5 \end{bmatrix}\right)$ as its columns. This will allow us to find the "directions" in which which neither of these vectors point. Such directions correspond precisely to the zero rows of this matrix in REF. Form the matrix

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 3 & 6 \\ 2 & -5 \end{pmatrix}.$$

Applying Gaussian elimination, we have

$$R_1 \leftarrow \frac{1}{2}R_1 : \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 3 & 6 \\ 2 & -5 \end{pmatrix}$$

$$R_1 \leftarrow \frac{1}{2}R_1 : \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 3 & 6 \\ 2 & -5 \end{pmatrix}$$

$$R_2 \leftarrow R_1 + R_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 6 \\ 2 & -5 \end{pmatrix}$$

$$R_3 \leftarrow -3R_1 + R_3 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 6 \\ 2 & -5 \end{pmatrix}$$

$$R_4 \leftarrow -2R_1 + R_4 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 6 \\ 0 & -5 \end{pmatrix}$$

$$R_3 \leftarrow -6R_2 + R_3 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 6 \\ 0 & -5 \end{pmatrix}$$

$$R_3 \leftarrow -6R_2 + R_3 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -5 \end{pmatrix}$$

$$R_4 \leftarrow 5R_2 + R_4 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, our missing directions correspond to the third and fourth slots in vectors in $\mathbb{R}^4$. Consequently, we form the following spanning list $\left(\left[\begin{smallmatrix} 2 \\ -1 \\ 3 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 6 \\ -5 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right] \right)$ in $\mathbb{R}^4$.

2. Reduce the list $\left(\left[\begin{smallmatrix} 3 \\ -2 \\ 6 \end{smallmatrix}\right], \left[\begin{smallmatrix} 4 \\ 1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \\ 6 \end{smallmatrix}\right], \left[\begin{smallmatrix} -2 \\ 1 \\ 1 \end{smallmatrix}\right] \right)$ to a linearly independent list in $\mathbb{R}^3$.

Solution. Note that no list of length greater than 3 in $\mathbb{R}^3$ can be linearly independent, so we know we will have to drop a vector from this list. We begin by forming the following matrix

$$\begin{pmatrix} 3 & 4 & 1 & -2 \\ -2 & 1 & 2 & 1 \\ 6 & 3 & 6 & 1 \end{pmatrix}.$$

We then apply Gaussian elimination to this matrix.

$$R_1 \leftarrow \frac{1}{3}R_1 : \begin{pmatrix} 1 & 4/3 & 1/3 & -2/3 \\ -2 & 1 & 2 & 1 \\ 6 & 3 & 6 & 1 \end{pmatrix}$$

$$R_2 \leftarrow R_2 + 2R_1 : \begin{pmatrix} 1 & 4/3 & 1/3 & -2/3 \\ 0 & 11/3 & 8/3 & -1/3 \\ 6 & 3 & 6 & 1 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - 6R_1 : \begin{pmatrix} 1 & 4/3 & 1/3 & -2/3 \\ 0 & 11/3 & 8/3 & -1/3 \\ 0 & -5 & 4 & 5 \end{pmatrix}$$

$$R_2 \leftarrow \frac{3}{11}R_2 : \begin{pmatrix} 1 & 4/3 & 1/3 & -2/3 \\ 0 & 1 & 8/11 & -1/11 \\ 0 & -5 & 4 & 5 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - \frac{4}{3}R_2 : \begin{pmatrix} 1 & 0 & -7/11 & -6/11 \\ 0 & 1 & 8/11 & -1/11 \\ 0 & -5 & 4 & 5 \end{pmatrix}$$

$$R_3 \leftarrow R_3 + 5R_2 : \begin{pmatrix} 1 & 0 & -7/11 & -6/11 \\ 0 & 1 & 8/11 & -1/11 \\ 0 & 0 & 84/11 & 50/11 \end{pmatrix}$$

$$R_3 \leftarrow \frac{11}{84}R_3 : \begin{pmatrix} 1 & 0 & -7/11 & -6/11 \\ 0 & 1 & 8/11 & -1/11 \\ 0 & 0 & 1 & 25/42 \end{pmatrix}$$

$$R_1 \leftarrow R_1 + \frac{7}{11}R_3 : \begin{pmatrix} 1 & 0 & 0 & -1/6 \\ 0 & 1 & 8/11 & -1/11 \\ 0 & 0 & 1 & 25/42 \end{pmatrix}$$

$$R_2 \leftarrow R_2 - \frac{8}{11}R_3 : \begin{pmatrix} 1 & 0 & 0 & -1/6 \\ 0 & 1 & 0 & -11/21 \\ 0 & 0 & 1 & 25/42 \end{pmatrix}.$$

The fourth column of the row-reduced matrix has no pivots in its last column. Consequently, the vector $\left[\begin{smallmatrix} -2 \\ 1 \\ 1 \end{smallmatrix}\right]$ is in the span of the vectors $\left(\left[\begin{smallmatrix} 3 \\ -2 \\ 6 \end{smallmatrix}\right], \left[\begin{smallmatrix} 4 \\ 1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \\ 6 \end{smallmatrix}\right]\right)$. We obtain the following linearly independent list: $\left(\left[\begin{smallmatrix} 3 \\ -2 \\ 6 \end{smallmatrix}\right], \left[\begin{smallmatrix} 4 \\ 1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \\ 6 \end{smallmatrix}\right]\right)$.

3. Extend the list $\left(\left[\begin{smallmatrix} 2 \\ -1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ -2 \end{smallmatrix}\right]\right)$ to a spanning list of $\mathbb{Z}_5^3$.

Solution. Note that no list of length 2 of vectors in $\mathbb{Z}_5^3$ can span $\mathbb{Z}_5^3$, thus we will have to add a third vector. We use Gaussian elimination on the following matrix to accomplish this. First, we form the following matrix:

$$\begin{pmatrix} 2 & 0 \\ 4 & 1 \\ 3 & 3 \end{pmatrix}.$$

Applying Gaussian elimination, we have

$$R_1 \leftarrow 3R_1 : \begin{pmatrix} 1 & 0 \\ 4 & 1 \\ 3 & 3 \end{pmatrix}$$

$$R_2 \leftarrow R_1 + R_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 3 \end{pmatrix}$$

$$R_3 \leftarrow 2R_1 + R_3 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}$$

$$R_3 \leftarrow 2R_2 + R_3 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Consequently, our missing direction, indicated by the zero row, is supplemented by the addition of the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the list. The list $\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ thus spans $\mathbb{Z}_5^3$, as the above Gaussian elimination demonstrates.

4. Provide two different spanning lists of $\mathbb{Z}_7^2$ and prove they span $\mathbb{Z}_7^2$.

Solution. The "simplest" spanning list of $\mathbb{Z}_7^2$ is

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

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This list is clearly spanning, since for any $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{Z}_7^2$, we have

$$\begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The next most simple spanning list is

$$\left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right).$$

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This list is spanning as well, since for any $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{Z}_7^2$, we have

$$\begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = -c_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + -c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + -c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

5. ($\star$) The vectors $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}\right)$ span a certain subspace of $\mathbb{R}^3$ (i.e., the trivial subspace, a line, a plane, or $\mathbb{R}^3$ itself). Determine the kind of subspace they span. Reduce them to a span-preserving linearly independent list (if the list is not linearly independent already).

Solution. Form the matrix

$$\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

We apply Gaussian elimination:

$$R_2 \leftarrow 2R_2 + R_1 : \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - R_2 : \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - R_2 : \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_1 \leftarrow \frac{1}{2}R_1 : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the list is linearly dependent. In particular, the third vector can be written as a linear combination of the first two. Thus, the given list of vectors spans a plane in $\mathbb{R}^3$. We can drop the third vector from the list and retain the span of the original list of vectors. Furthermore, dropping the vector yields a linearly independent list by the Gaussian elimination above. The final list is $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$.

2 Bases and Dimension

1. Is the list $\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right)$ a basis for $\mathbb{R}^3$? Justify your response.

Solution. The list *does* form a basis for $\mathbb{R}^3$. We form the matrix

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

Applying Gaussian elimination, we have

$$R_2 \leftarrow R_2 - 2R_1 : \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 2 & 2 & 1 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - 2R_1 : \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & -2 & -3 \end{pmatrix}$$

$$R_3 \leftarrow 3R_3 - 2R_2 : \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{pmatrix}$$

$$R_3 \leftarrow -\frac{1}{5}R_3 : \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \leftarrow R_2 + 2R_3 : \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - 2R_3 : \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \leftarrow -\frac{1}{3}R_2 : \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - 2R_2 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Consequently, the list does form a basis of $\mathbb{R}^3$, since the reduced matrix possesses a pivot in every row (spanning) and a pivot in every column (linearly independent).

2. Extend the list $\left(\left[\begin{smallmatrix} 2 \\ -7 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 5 \\ -3 \\ 1 \end{smallmatrix}\right]\right)$ to a basis for $\mathbb{R}^3$. Prove that the resulting list is a basis.

Solution. We identify the zero row via Gaussian elimination. First, we form the matrix

$$\begin{pmatrix} 2 & 5 \\ -7 & -3 \\ 3 & 1 \end{pmatrix}.$$

Then, applying Gaussian elimination, we find

$$R_2 \leftarrow 2R_2 + 7R_1 : \begin{pmatrix} 2 & 5 \\ 0 & 29 \\ 3 & 1 \end{pmatrix}$$

$$R_3 \leftarrow 2R_3 - 3R_1 : \begin{pmatrix} 2 & 5 \\ 0 & 29 \\ 0 & -13 \end{pmatrix}$$

$$R_2 \leftarrow \frac{1}{29}R_2 : \begin{pmatrix} 2 & 5 \\ 0 & 1 \\ 0 & -13 \end{pmatrix}$$

$$R_3 \leftarrow R_3 + 13R_2 : \begin{pmatrix} 2 & 5 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - 5R_2 : \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R_1 \leftarrow \frac{1}{2}R_1 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, the two vectors are linearly independent. We have a zero row in the third row, so the two vectors are spanning (though it should have already clear that we need to append a third vector, since no list of length 2 can be spanning in $\mathbb{R}^3$). We append the vector $\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$ to our original list, obtaining the list of vectors $\left(\left[\begin{smallmatrix} 2 \\ -7 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 5 \\ -3 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]\right)$. This list is linearly independent by the above Gaussian elimination, and is spanning by the same sort of argument.

3. Provide a basis for $\mathbb{C}^2$ that is not a scalar multiple of the standard basis. Prove that it is a basis for $\mathbb{C}^2$.

Solution. A simple basis which is not a scalar multiple of the standard basis is

$$\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

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We form the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Applying Gaussian elimination to this matrix, we have

$$R_2 \leftarrow R_1 + R_2 : \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$R_2 \leftarrow \frac{1}{2} R_2 : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - R_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The resulting matrix has a pivot in every row and a pivot in every column. Consequently, the list

$$\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

is a basis of $\mathbb{C}^2$.

4. What is the dimension of $M_{2,4}(\mathbb{R})$? Justify your response.

Solution. The dimension of $M_{2,4}(\mathbb{R})$ is 8. Consider can arbitrary element of $M_{2,4}(\mathbb{R})$. We have

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \cdots + a_8 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The list of matrices

$$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

of length 8 thus clearly forms a basis of $M_{2,4}(\mathbb{R})$, and consequently $M_{2,4}(\mathbb{R})$ has dimension 8.

5. Modify the list $\left(\left[\begin{smallmatrix} -1 \\ 3 \end{smallmatrix} \right], \left[\begin{smallmatrix} 2 \\ -4 \end{smallmatrix} \right], \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right] \right)$ so that the resulting list is a basis for $\mathbb{Z}_7^2$. Prove that the resulting list is a basis.

Solution. We first form the matrix

$$\begin{pmatrix} -1 & 2 & 3 \\ 3 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}.$$

Applying Gaussian elimination, we have

$$R_1 \leftarrow 6, R_1 : \begin{pmatrix} 1 & 5 & 4 \\ 3 & 3 & 1 \end{pmatrix}.$$

$$R_2 \leftarrow R_2 - 3R_1 : \begin{pmatrix} 1 & 5 & 4 \\ 0 & 2 & 3 \end{pmatrix}$$

$$R_2 \leftarrow 4, R_2 : \begin{pmatrix} 1 & 5 & 4 \\ 0 & 1 & 5 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - 5R_2 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix}$$

Thus, the reduced matrix has no pivots in its third column. Consequently, the corresponding vector can be written as a linear combination of the previous two corresponding vectors, and the list of vectors is currently linearly dependent. Dropping this vector from the list, we obtain a linearly independent list that is also spanning by the Gaussian elimination above. Consequently, the list $\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}\right)$ is a basis for $\mathbb{Z}_7^2$.

6. Let $\mathbb{F}$ be a field. Provide a basis for $\mathbb{F}$. Prove that it is a basis for $\mathbb{F}$. Are there lists of elements in $\mathbb{F}$ which do not comprise a basis?

Solution. Let 1 denote the multiplicative identity of $\mathbb{F}$. Clearly, the list (1) is linearly independent in $\mathbb{F}$ (fields do not have non-zero elements a that would allow for $a \cdot 1 = 0$). The list (1) also spans $\mathbb{F}$, since for any $x \in \mathbb{F}$, $x \cdot 1 = x \in \mathbb{F}$. Thus, (1) is a basis for $\mathbb{F}$.

There is exactly one element of $\mathbb{F}$ which does not form a basis for $\mathbb{F}$: the additive identity of $\mathbb{F}$, 0, since $x \cdot 0 = 0$ for all $x \in \mathbb{F}$.

7. Extend the list $\left(\begin{bmatrix} 1 \\ -4 \\ 5 \\ 2 \end{bmatrix}\right)$ to a basis for $\mathbb{R}^4$. Prove that the resulting list is a basis.

Solution. Clearly this list is not itself a basis for $\mathbb{R}^4$, since it has length less than 4 and thus cannot be spanning. A good candidate basis for problems in which we only begin with one vector is to append elements of the standard basis and check for linear independence and spanning. We form the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix has a pivot in every row and in every column. Consequently, the list $\left(\begin{bmatrix} 1 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$ spans $\mathbb{R}^4$ and is linearly independent in $\mathbb{R}^4$, and is thus a basis for $\mathbb{R}^4$.

8. Show that the same three vectors from Exercise 1 form a basis of $\mathbb{Z}_2^3$.

Solution. Form the matrix

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

Our field is $\mathbb{Z}_2$, and $0 = 2 \mod 2$. Thus, this matrix is equal to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which clearly implies that the list of vectors forms as a basis for $\mathbb{Z}_2^3$.

9. Provide a list of vectors that is a basis for $\mathbb{R}^2$, but not for $\mathbb{Z}_5^2$. Prove that it is a basis for $\mathbb{R}^2$, but not for $\mathbb{Z}_5^2$

Solution. We can leverage the properties of arithmetic modulo 5. Consider the list of vectors $\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]\right)$ in $\mathbb{R}^2$. Applying Gaussian elimination to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}$$

having these vectors as its columns, we find

$$R_1 \leftarrow -\frac{1}{5} R_2 + R_1 : \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$R_2 \leftarrow \frac{1}{5} R_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, these vectors form a basis for $\mathbb{R}^2$. However, when considering these vectors in $\mathbb{Z}_5^2$, we find that, modulo 5, we have the equality

$$\begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

which implies that the two vectors are linearly dependent, and thus cannot comprise a basis for $\mathbb{Z}_5^2$.

10. ($\star$) A finite dimensional vector space cannot contain an infinite dimensional subspace, but can an infinite dimensional vector space contain a finite dimensional subspace?

Solution. Yes. As a simple example, consider the vector space of all real sequences:

$$\mathbb{R}^\infty = \{(a_n)_{n=1}^\infty : a_n \in \mathbb{R}\}$$

with vector addition and scalar multiplication defined point-wise. Clearly this vector space is infinite dimensional, since there is no finite spanning list of vectors in this space. However, given the subspace

$$U = \{(a_n)_{n=1}^\infty \in \mathbb{R}^\infty : a_n = 0 \text{ for } n > 1\}$$

the vector $(1, 0, 0 \ldots) \in U$ forms a linearly independent list and spans U. Thus, U is a finite-dimensional subspace of an infinite-dimensional vector space.

11. ($\star$) Show that the list $\left(\begin{bmatrix} k-1 \\ k \\ k \\ k \end{bmatrix}, \begin{bmatrix} k \\ k-1 \\ k \\ k \end{bmatrix}, \begin{bmatrix} k \\ k \\ k-1 \\ k \end{bmatrix}, \begin{bmatrix} k \\ k \\ k \\ k-1 \end{bmatrix}\right)$, $k \in \mathbb{Z}$, $k \geq 1$, is a basis for $\mathbb{R}^4$.

Solution. Label the vectors in the list v_1, v_2, v_3, v_4 respectively. Note the following equalities:

$$v_2 - v_1 = \begin{bmatrix} k \\ k-1 \\ k \\ k \end{bmatrix} - \begin{bmatrix} k-1 \\ k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

$$v_3 - v_1 = \begin{bmatrix} k \\ k \\ k-1 \\ k \end{bmatrix} - \begin{bmatrix} k-1 \\ k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$v_4 - v_1 = \begin{bmatrix} k \\ k \\ k \\ k-1 \end{bmatrix} - \begin{bmatrix} k-1 \\ k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

The vectors given by the differences are linearly independent. Furthermore, v_1 is not in the span of these vectors. Consequently, appending v_1 to this list of vectors retains the linear independence of the list. Thus, these vectors yield a linear independent list of length 4 in $\mathbb{R}^4$, giving a basis. Adding v_1 back to each vector in this list gives the desired conclusion.

Note that one could have also answered this question via direct Gaussian elimination, but the "brute force" solution is often not optimal in terms of time spent.

12. True or false: there is a spanning list of length 2 in $\mathbb{R}^4$. Justify your response.

Solution. False. For any vector space $\mathbb{F}^n$, all spanning lists must have length at least n. In this specific case, a list of $2 \leq 4$ cannot span $\mathbb{R}^4$.

13. True or false: there is a linearly independent list of length 9 in $\mathbb{R}^{15}$. Justify your response.

Solution. True. Consider the list comprising of the first nine elements of the standard basis of $\mathbb{R}^{15}$.