Vector Subspaces, Span, and Linear Independence

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Contents

1 Vector (sub)spaces

1.1 Vector Spaces

We have already observed that vector spaces are "well-behaved." In particular, they are closed under addition and scalar multiplication, and these operations behave as expected (i.e., they are commutative, associative, and possess distributive properties). Formally:

Let V be a vector space over the field $\mathbb{F}$. Then the following properties hold.

1. (**Commutativity.**) For all $x, y \in V$, x + y = y +x
2. (**Associativity.**) For all $x, y, z \in V$, (x + y) + z = x +(y + z)
3. (**Additive identity.**) There exists a unique element $0 \in V$ such that for all $v \in V$, $v + 0 = 0 + v = v$.
4. (**Closure under addition.**) For all $x, y \in V$, $x + y \in V$.
5. (**Additive inverse.**) For all $x \in V$, there exists a unique $y \in V$ such that $x + y = y + x = 0$. We denote this additive inverse y by $-x$.

The following properties concern the field $\mathbb{F}$ and its interaction with V. Note that fields are guaranteed to have unique additive and multiplicative inverses, as well as unique additive and multiplicative identities, typically denoted by 0 and 1 respectively.

6. (**Closure under scalar multiplication.**) For all $x \in V$ and for all $a \in \mathbb{F}$, $ax \in V$.
7. (**Multiplicative identity**.) For all $x \in V$, $1 \cdot x = x \cdot 1 = x$.
8. (**Multiplicative associativity.**) For all $a, b \in \mathbb{F}$ and for all $x \in V$, $a(bv) = (ab)v$.
9. (**Distributivity.**) For all $x, y \in V$ and for all $a, b \in \mathbb{F}$, $a(x+y) = ax + ay$ and $(a+b)(x) = ax + bx$.

Elements of V are called *vectors*. Elements of $\mathbb{F}$ are called *scalars*.
If you do not recall the formal definition of a field, do not worry; a field is simply a set equipped with multiplication and addition operations that behave nicely together (i.e., distributivity, commutativity, associativity, existence of 0, 1, and the existence of additive/multiplicative inverses for every element in the set). You can think of the properties of $\mathbb{R}$ or $\mathbb{Q}$ as examples. Note that a field need not be infinite (e.g., $\mathbb{Z}_3$, the integers modulo 3).

It is natural to think of the field $\mathbb{F}$ as "acting"[1] on V. Concretely, this means that the addition on V comes from "within" V itself, while the scalar multiplication comes from $\mathbb{F}$. This is why we were able discuss the properties of addition on V prior to considering $\mathbb{F}$. It is also why Property 8 above allows us to multiply scalars in $\mathbb{F}$ prior to involving vectors at all.

In general, scalar multiplication, vector addition, and their interaction heavily depend on what $\mathbb{F}$ and V are and how we wish for them to behave together. In this course, all of our vector spaces will be "the same"[2] as $\mathbb{F}^n$, so scalar multiplication and vector addition are defined entry-wise.

Remark 1.1. *The term "scalar" comes from the Latin term "scalaris," which describes the attribute of belonging to a ladder (a "scala"). Vector derives from the Latin verb "vehere," which means to possess or to carry.*

We now discuss a few examples

Example 1.1 ($\mathbb{R}^n$)**.** *Consider the set of column n-tuples $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $a_i \in \mathbb{R}$, $1 \le i \le n$ with addition and scalar multiplication defined entry-wise. This set with these operations over the scalar field $\mathbb{R}$ form the vector space $\mathbb{R}^n$. We exhibit these operations explicitly. Let $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $u = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, with the a_i and b_i in $\mathbb{R}$. Let $c \in \mathbb{R}$ be a scalar. Then*

[1]The word "action" has a precise mathematical meaning here. For the curious, see "module over a field."

[2]The phrase "the same" here refers to the term "vector space isomorphism."

$$v + u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

and

$$c \cdot v = c \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}.$$

Remember to be aware of where operations are taking place. In Example 1.1, the sum $v + u$ uses the addition defined on $\mathbb{R}^n$ between *vectors*, while the sum $a_i + b_i$ in each entry uses the addition between *scalars* of the field $\mathbb{R}$.

Example 1.2 ($M_2(\mathbb{Z}_5)$)**.** *Denote by $M_2(\mathbb{Z}_5)$ the set of 2×2 matrices with entries in $\mathbb{Z}_5$ (the integers modulo 5). Define addition and scalar multiplication of these matrices entry-wise, with the scalar field being $\mathbb{Z}_5$.*

We check closure under addition and scalar multiplication here; the verification of the rest of the vector space properties (e.g., unique additive identity, distributivity, and so on) are left as an exercise.

Consider arbitrary matrices $A, B \in M_2(\mathbb{Z}_5)$,

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

Taking their sum, which is defined by summing entry-wise, we see

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}.$$

Since each entry is a sum of elements in $\mathbb{Z}_5$, each entry is also in $\mathbb{Z}_5$ (recall that this holds because $\mathbb{Z}_5$ is a field, and fields are closed under addition). Thus, $M_2(\mathbb{Z}_5)$ is closed under addition.

Similarly, consider a scalar $c \in \mathbb{Z}_5$. Then

$$c \cdot A = \begin{bmatrix} c \cdot a_1 & c \cdot a_2 \\ c \cdot a_3 & c \cdot a_4 \end{bmatrix}.$$

The entries $c \cdot a_i$, $1 \leq i \leq 4$ of this matrix are products of elements of $\mathbb{Z}_5$, and $\mathbb{Z}_5$ is closed under multiplication, so $c \cdot A$ is also in $M_2(\mathbb{Z}_5)$.

Note that Examples 1.1 and 1.2 generalize. Specifically, if $\mathbb{F}$ is any field and $n \geq 1$, then $\mathbb{F}^n$ over $\mathbb{F}$ is always a vector space with the operations defined entry-wise. Similarly, $M_n(\mathbb{F})$ over $\mathbb{F}$ is always a vector space with the operations defined entry-wise. In particular, $\mathbb{F}^1 = \mathbb{F}$ is a vector space. That is, fields are vector spaces (but the converse is not necessarily true).

In general, the verification of whether a given set with scalar multiplication and addition is a vector space or not is time-consuming given the number of properties to check, but quick checks can disqualify a set from being a vector space. For example, if scalar multiplication does not use a field (e.g., $\mathbb{Z}$), then the set with such a scalar multiplication defined on it cannot possibly be a vector space.

1.2 Vector Subspaces

In many cases, understanding the structure of certain subsets of a vector space can aid in understanding the entire space. Heuristically, it makes sense that isolating a smaller or "simpler" part of a larger object, still retaining all of the relevant properties of the larger object, should aid in understanding the structure of the larger whole. This idea appears in many places in mathematics, but in our setting, it manifests as the notion of a *vector subspace*.

Formally, a vector subspace is the following.

Definition 1.1 (Vector subspace)**.** *Let V be a vector space over the field $\mathbb{F}$. Let $S \subset V$, $S \neq \varnothing$. Suppose that S has the following properties:*

1. *For all $x, y \in S$, $x + y \in S$.*
2. *For all $x \in S$ and for all $c \in \mathbb{F}$, $c \cdot x \in S$.*

Then we call S a vector subspace of V, or simply a subspace of V when the vector space setting is understood.

Remark 1.2. *Some authors add the additional requirement that $0 \in S$. In the vector space setting, this is redundant. Indeed, choosing $0 \in \mathbb{F}$ and some $x \in V$, we have $x \cdot 0 = 0 \in S$ by the second property.*

We provide a criterion for a subset of a vector space to be a vector subspace.

Proposition 1.1 (Subspace criterion)**.** *Let V be a vector space over the field $\mathbb{F}$. Then $S \subset V$, $S \neq \varnothing$, is a subspace of V if and only if for all $x, y \in S$ and for all $c \in \mathbb{F}$, $x + cy \in S$.*

Proof. First, suppose S is a subspace of V. Choose $x, y \in S$ and $c \in \mathbb{F}$. Then $cy \in S$ since subspaces are closed under scalar multiplication, and $x + cy \in S$ since vector spaces are closed under addition. Now suppose S satisfies the subspace criterion. Choose again $x, y \in S$. Selecting $c = 1$, we have $x + y \in S$, which gives us the first condition in the definition of a subspace. Next, choosing $x = 0$, we have $cy \in S$, giving us the second condition. $\square$

For every vector space V, we can immediately identify two subspaces of V. Firstly, $V \subset V$ is a subspace of V. Secondly, one can (and should) verify that $\{0\} \subset V$ is a subspace of V. The latter subspace is called the *trivial subspace* of V.

Remark 1.3. *In Definition 1.1 and Proposition 1.1, we specified that $S \neq \varnothing$. One might wonder why this was necessary. This condition is actually necessitated by the definition of a vector space. We require S to possess a unique additive identity 0, but if S were empty, no such element would exist in S. Henceforth, we shall assume that such S are nonempty, bearing this technicality in mind.*

We conclude this section with examples and non-examples of vector subspaces.

Example 1.3. *Consider the vector space $\mathbb{R}^3$ and the set*

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_3 = 0 \right\}.$$

The set S is a subspace of $\mathbb{R}^3$. Indeed, let $v = \left[\begin{smallmatrix} a_1 \\ a_2 \\ 0 \end{smallmatrix}\right] \in S$ and $u = \left[\begin{smallmatrix} b_1 \\ b_2 \\ 0 \end{smallmatrix}\right] \in S$. We have

$$v + u = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ 0 \end{bmatrix} \in S.$$

We verify closure under scalar multiplication similarly. Let $c \in \mathbb{R}$. Then

$$c \cdot v = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot a_2 \\ 0 \end{bmatrix} \in S.$$

Consequently, S is a subspace of $\mathbb{R}^3$.

Example 1.4. *Consider the first octant of $\mathbb{R}^3$ defined by*

$$U = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_1, a_2, a_3 \geq 0 \right\}.$$

This set is not a subspace of $\mathbb{R}^3$. Setting $v = \left[\begin{smallmatrix} a_1 \\ a_2 \\ a_2 \end{smallmatrix}\right] \in S$ and $u = \left[\begin{smallmatrix} b_1 \\ b_2 \\ b_3 \end{smallmatrix}\right] \in S$, we have

$$v + u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

Note that $v + u$ is in S, so S is closed under addition. However, choose $c \in \mathbb{R}$ with $c < 0$. Then

$$c \cdot v = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}.$$

Thus $c \cdot v$ is not in U, since all entries of $c \cdot v$ are negative for choices of negative c. Consequently, U is not closed under scalar multiplication, and is not a subspace of $\mathbb{R}^3$.

Example 1.5. *Consider the vector space $\mathbb{C}^3$ and the set*

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{C}^3 : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

The set S is not a vector subspace of $\mathbb{C}^3$. Indeed, choose $i \in \mathbb{C}$. Let $v = \left[\begin{smallmatrix} a_1 \\ a_2 \\ a_2 \end{smallmatrix}\right] \in S$. Then

$$i \cdot v = i \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 i \\ a_2 i \\ a_3 i \end{bmatrix} \notin S.$$

Consequently, S fails to be closed under scalar multiplication, and is thus not a subspace of $\mathbb{C}^3$.

Example 1.6. *Consider the vector space $\mathbb{R}^2$ and the subset*

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_1 = 2a_2 \right\}.$$

The subspace criterion proves useful here. Consider $v = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] \in S$ and $u = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \in S$. Let $c \in \mathbb{R}$. Then

$$v + cu = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + c \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2a_2 \\ a_2 \end{bmatrix} + \begin{bmatrix} 2cb_2 \\ cb_2 \end{bmatrix} = \begin{bmatrix} 2(a_2 + cb_2) \\ a_2 + cb_2 \end{bmatrix} \in S.$$

Thus, by Proposition 1.1, S is a subspace of $\mathbb{R}^2$.

1.3 Exercises

Exercises that require more creativity or deeper reasoning are marked with a $\star$. The bonus question is simply for fun, and does not concern required material for this course.

1. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_1 = -a_2 \right\}$$
a subspace of $\mathbb{R}^2$? Justify your answer with rigorous reasoning.

2. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = a_1^2 \right\}$$
a subspace of $\mathbb{R}^2$? Justify your answer with rigorous reasoning.

3. ($\star$) Describe a nontrivial subspace of $\mathbb{C}^2$ over the field $\mathbb{R}$ that is not simply the restriction of some entries to 0. Prove it is a subspace.

4. Choose your favorite vector
$$v = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2.$$
Can you describe a nontrivial vector subspace of $\mathbb{R}^2$ containing v?

5. ($\star$) Show that every line passing through the origin of $\mathbb{R}^2$ (i.e., containing the zero vector) is a subspace of $\mathbb{R}^2$.

6. Why is $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{Z}_5^2 : a_1, a_2 \text{ are odd modulo } 5 \right\}$ not a subspace of $\mathbb{Z}_5^2$?

7. Let $\mathbb{F}$ be a field (and thus a vector space). What are the subspaces of $\mathbb{F}$?

8. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_3 = a_1 - a_2 \right\}$$
a subspace of $\mathbb{R}^3$? Justify your answer with rigorous reasoning.

9. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_2 = 4a_3, a_1 = -2a_2 \right\}$$
a subspace of $\mathbb{R}^3$? Justify your answer with rigorous reasoning.

10. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_1 + 2a_2 + 4a_3 = 5 \right\}$$
a subspace of $\mathbb{R}^3$? Justify your answer with rigorous reasoning.

Bonus. Let
$$P_3(x) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$
be the set of polynomials of degree at most 3 in x with real coefficients. This set is a vector space with addition and scalar multiplication defined the usual way. Show that the set $S = \{p \in P_3(x) : \text{p has degree at most } 1\}$ is a subspace of $P_3(x)$.

2 Span and Linear Independence

2.1 Span

We have witnessed many examples of vector subspaces, but these were already provided to us. One might ask how vector subspaces arise naturally from a given vector space. This leads us to the notion of linear combinations and span.

Definition 2.1 (Linear combination)**.** *Let V be a vector space over the field $\mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ be a finite list of vectors in V. A linear combination of $\{v_1, \ldots, v_n\}$ is a vector of the form*

$$c_1 v_1 + \cdots + c_n v_n,\ c_i \in \mathbb{F},\ 1 \leq i \leq n.$$

Intuitively, one can picture a linear combination vector as a weighted sum describing "how much" of each vector in the list it "contains." For example, consider the list $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$ of vectors in $\mathbb{R}^2$. The linear combination

$$5 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 7 \cdot \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$$

can be interpreted as describing a vector containing five instances of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and seven instances of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Since these vectors are in $\mathbb{R}^2$, we can visualize their linear combination above as five steps along the horizontal axis of the plane and seven steps along the vertical axis of the plane.

Remark 2.1. *While visual-spatial reasoning can be helpful for building base intuition or for reasoning about problems in $\mathbb{R}/\mathbb{R}^2/\mathbb{R}^3/\mathbb{C}$, it is important not to rely on it in general, as there are very few vector spaces which can be visualized in the fashion above*[3]*.*

Naturally, we would like to see if, given such a list of vectors in a vector space, the set of all such linear combinations of that list has any sort of structure. This leads us to the following definition and proposition.

Definition 2.2 (Span)**.** *Let V be a vector space over the field $\mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ be a finite list of vectors in V. We define*

$$span\{v_1, \ldots, v_n\} := \{c_1 v_1 + \cdots + c_n v_n \in V : c_i \in \mathbb{F},\ 1 \leq i \leq n\}.$$

In other words, the span of a list of vectors is the set of all linear combinations of the vectors in that list. As an immediate consequence of this definition, we have the following proposition

Proposition 2.1. *Let V be a vector space. Let $\{v_1, \ldots, v_n\}$ be a list of vectors in V. Then $span\{v_1, \ldots v_n\}$ is a subspace of V. In particular, it is the smallest*[4] *subspace of V containing $\{v_1, \ldots, v_n\}$.*

[3] There are creative methods for visualizing certain subsets of spaces like $\mathbb{C}^2$. See "domain coloring" for more.

[4] There are many ways to be "small" in mathematics. In this case, we mean inclusion-wise. That is, every subspace S of V containing $\{v_1, \ldots, v_n\}$ also contains $\text{span}\{v_1, \ldots v_n\}$.

The proof of this proposition is left as an exercise, as it is obtained from a direct application of the definitions of subspace and span. We now provide examples of vector spaces obtained from spans of lists of vectors.

Example 2.1. *Consider the list* $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ *of vectors in* $\mathbb{R}^3$*, and consider an element* $v \in$ *span*$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$*. The vector* v *is of the form*

$$v = c_1 \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}, \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

Notice that the set of all such v *is exactly the set of vectors* $\mathbb{R}^3$*. Consequently, we say that* $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ spans $\mathbb{R}^3$.

The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$ are called the *standard basis vectors* for $\mathbb{R}^3$, and they are denoted by e_1, e_2, and e_3. The notation and terminology for $\mathbb{R}^n$ is analogous. We will return to the term "basis" later.

Intuitively, one can view e_1, e_2, and e_3 as representing the directions of the standard coordinate axes in $\mathbb{R}^3$; any vector in $\mathbb{R}^3$ can be obtained by "moving" the appropriate distance along each coordinate axis.

Example 2.2. *Consider the list* $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ *of vectors in* $\mathbb{R}^3$*, and consider an element* $v \in$ *span*$\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$*. The vector* v *is of the form*

$$v = c_1 \cdot \begin{bmatrix}1\\0\\2\end{bmatrix} + c_2 \cdot \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\2c_1 + c_2\end{bmatrix}.$$

The collection of all such vectors forms a plane in $\mathbb{R}^3$*.*

So far, we have asked what the span of a list of vectors looks like as a subspace, and what kinds of vectors are in that subspace. A natural question arises in the other direction: given a list of vectors $\{v_1, \ldots, v_n\}$ in a vector space V over the field $\mathbb{F}$, how can we determine when a vector $u \in V$ is in $\operatorname{span}\{v_1, \ldots, v_n\}$? Gaussian elimination proves itself to be an effective tool to answer this question.

Write

$$u = \begin{bmatrix}u_1\\\vdots\\u_n\end{bmatrix}, \qquad v_i = \begin{bmatrix}v_i^{(1)}\\\vdots\\v_i^{(n)}\end{bmatrix}$$

for $u \in V$ and $v_i \in \{v_1, \ldots, v_n\} \subset V$. If $u \in \operatorname{span}\{v_1, \ldots, v_n\}$, then

$$u = \begin{bmatrix}u_1\\\vdots\\u_n\end{bmatrix} = \begin{bmatrix}v_1^1 c_1 + v_2^1 c_2 + \cdots + v_n^1 c_n\\\vdots\\v_1^n c_1 + v_2^n c_2 + \cdots + v_n^n c_n\end{bmatrix}$$

for some choice of scalars $c_i \in \mathbb{F}$, $1 \le i \le n$. Rewriting the latter matrix as

$$\begin{bmatrix} v_1^1 c_1 + v_2^1 c_2 + \cdots + v_n^1 c_n \\ \vdots \\ v_1^n c_1 + v_2^n c_2 + \cdots + v_n^n c_n \end{bmatrix} = A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

where A is the coefficient matrix of the v_i^j, we see that asking if $u \in \operatorname{span}\{v_1, \dots, v_n\}$ is the same as asking whether the system of equations described by

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

has a solution.

Example 2.3. *Consider the list of vectors* $\left\{ \left[\begin{smallmatrix} 5 \\ 0 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -10 \\ 0 \\ -2 \end{smallmatrix}\right] \right\}$ *in* $\mathbb{R}^3$. *Is* $\left[\begin{smallmatrix} 3 \\ 2 \\ -5 \end{smallmatrix}\right] \in$ *span*$\left\{ \left[\begin{smallmatrix} 5 \\ 0 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -10 \\ 0 \\ -2 \end{smallmatrix}\right] \right\}$*?*

Constructing the augmented matrix for this system, we obtain

$$\left(\begin{array}{ccc|c} 5 & 0 & -10 & 3 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & -2 & -5 \end{array}\right)$$

.

Gaussian elimination yields

$$R_1 \leftarrow R_1 - 2R_3 : \left(\begin{array}{ccc|c} 1 & 0 & -6 & 13 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & -2 & -5 \end{array}\right)$$

$$R_3 \leftarrow R_3 - 2R_1 : \left(\begin{array}{ccc|c} 1 & 0 & -6 & 13 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 10 & -31 \end{array}\right)$$

$$R_3 \leftarrow \tfrac{1}{10} R_3 : \left(\begin{array}{ccc|c} 1 & 0 & -6 & 13 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -\frac{31}{10} \end{array}\right)$$

$$R_1 \leftarrow R_1 + 6R_3 : \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{28}{5} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -\frac{31}{10} \end{array}\right)$$

.

Consequently, $\left[\begin{smallmatrix} 3 \\ 2 \\ -5 \end{smallmatrix}\right] \in$ *span*$\left\{ \left[\begin{smallmatrix} 5 \\ 0 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -10 \\ 0 \\ -2 \end{smallmatrix}\right] \right\}$*, with*

$$\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} = -28/5 \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 31/10 \cdot \begin{bmatrix} -10 \\ 0 \\ -2 \end{bmatrix}.$$

2.2 Linear Independence

A natural question is whether a vector in the span of a list of vectors can be uniquely represented as a linear combinaton of the list of vectors. This leads us to the following notion.

Definition 2.3 (Linearly (in)dependent list). *Let V be a vector space over the field $\mathbb{F}$. Let $(v_1, \dots, v_n)$ be a list of vectors in V. The list $(v_1, \dots, v_n)$ is said to be linearly independent if the only choice of $a_1, \dots, a_n \in \mathbb{F}$ such that*

$$a_1 v_1 + \cdots + a_n v_n = 0$$

is $a_1 = \cdots = a_n = 0$.

If $(v_1, \dots, v_n)$ is not linearly independent, it is said to be linearly dependent.

Remark 2.2. *Note that the empty list is considered to be linearly independent.*

Example 2.4. *Consider the list of vectors $(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}7\\0\end{smallmatrix}\right])$ in $\mathbb{R}^2$. These vectors are linearly dependent, since*

$$\begin{bmatrix}0\\0\end{bmatrix} = -7/2 \cdot \begin{bmatrix}2\\0\end{bmatrix} + 1 \cdot \begin{bmatrix}7\\0\end{bmatrix},$$

and thus there is a non-zero choice of c_1, c_2 such that the linear combination of these vectors with these coefficients is zero.

The definition of linear independence leads us to the following proposition

Proposition 2.2 (Uniqueness of coefficients). *Let V be a vector space over the field $\mathbb{F}$, and let $(v_1, \dots, v_n)$ be a linearly independent list of vectors in V. Let $v \in V$. Then there exists a unique choice of $c_1, \dots, c_n \in \mathbb{F}$ such that*

$$v = c_1 v_1 + \cdots + c_n v_n.$$

The proof of this proposition is once again left as an exercise, and you are highly encouraged to attempt the proof. As a hint, we remind the reader that most proofs concerning uniqueness of algebraic objects proceed by contradiction.

The process for determining whether a given list of vectors is linearly independent is analogous to that of determining whether a given vector rests in the span of a list of vectors. Let V be a vector space over the field $\mathbb{F}$. Let $\{v_1, \dots, v_n\}$ be a list of vectors in V, and write

$$v_i = \begin{bmatrix} v_i^{(1)} \\ \vdots \\ v_i^{(n)} \end{bmatrix},\ v_i^j \in \mathbb{F},\ 1 \le i, j \le n.$$

Asking if the list of the v_i is linearly independent is equivalent to asking if the the only choice of $c_1, \dots, c_n \in \mathbb{F}$ such that the following equality holds is $c_1 = \cdots = c_n = 0$.

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} v_1^1 c_1 + v_2^1 c_2 + \cdots + v_n^1 c_n \\ \vdots \\ v_1^n c_1 + v_2^n c_2 + \cdots + v_n^n c_n \end{bmatrix}.$$

Intuitively[5], a list of vectors is linear independent when each vector in the list contributes in a "direction" that the other vectors do not. In the case of Example 2.4, both vectors only contributed or "pointed" horizontally, and neither vertically, and consequently they were linearly dependent. While this visual intuition breaks down in spaces with more than three entries, the algebraic intuition (i.e., observing how each vector contributes to each entry) holds.

We now provide an example of a linearly independent list of vectors and note how to identify linear independence in such simple cases.

Example 2.5. *Consider the list of vectors* $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ 0 \end{bmatrix}\right\}$ *in* $\mathbb{R}^3$. *Before immediately applying Gaussian elimination, we should use our intuition to form an understanding of this list. Immediately, we can see that, since only the second vector has a non-zero second entry, its coefficient in any linear combination of the list equaling zero must be itself zero.*

Thus, we now need only consider the first and third vectors in the list. In a linear combination of these vectors totaling zero, we require that the third entry of the combination is also zero, and this is only achieved when the first vector in this list has coefficient zero in the linear combination.

This leaves us with the question of whether any non-zero scalar makes the third vector zero, and the answer to that question is clearly no.

Thus, all coefficients are forced to be zero, and the list is linearly independent.

Note that in Example 2.5, we determined linear independence merely using our intuition and knowledge of linear independence. This solution saved us time, and if written carefully, would constitute an adequately reasoned solution. In sum, we can leverage our algebraic intuition to determine linear independence for sufficiently "small"[6] vector spaces. Note that this approach would be ill-suited for "larger" spaces, say, $\mathbb{R}^{10^{100}}$.

2.3 Exercises

1. Is the list of vectors

$$\left\{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 4 \end{bmatrix}\right\}$$

   in $\mathbb{R}^3$ linearly independent? Justify your response.

[5] We will formalize this intuition next week when we discuss dimension.

[6] The question of what it means for a space to small is actually quite broad. In linear algebra, we often mean "algebraic dimension," in which case $\mathbb{Q}^3$ and $\mathbb{R}^3$ are the same "size." However, if we consider size from a "volume" perspective, $\mathbb{Q}^3$ has volume 0, and is small in $\mathbb{R}^3$. If we consider size from a distance perspective, $\mathbb{Q}^3$ is suddenly large in $\mathbb{R}^3$ again, and actually "fills" $\mathbb{R}^3$.

2. Is the list of vectors
$$\left\{\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}4\\2\\1\end{bmatrix},\begin{bmatrix}0\\2\\4\end{bmatrix}\right\}$$
in $\mathbb{Z}_5^3$ linearly independent? Justify your response.

3. Describe the set
$$\operatorname{span}\left\{\begin{bmatrix}-2\\1\\-3\end{bmatrix},\begin{bmatrix}5\\1\\2\end{bmatrix}\right\}.$$
What geometric figure in $\mathbb{R}^3$ does it form?

4. Is
$$\begin{bmatrix}0\\4\\1\end{bmatrix}\in\operatorname{span}\left\{\begin{bmatrix}-2\\1\\4\end{bmatrix},\begin{bmatrix}3\\3\\2\end{bmatrix}\right\},$$
where all vectors are in $\mathbb{Z}_5^3$? Justify your response.

5. Identify a vector v in $\mathbb{R}^3$ such that when v is added to the list
$$\left\{\begin{bmatrix}6\\-5\\7\end{bmatrix}\right\},$$
the resultant list is linearly independent.

6. Is
$$\begin{bmatrix}4\\3\\-5\\1\end{bmatrix}\in\operatorname{span}\left\{\begin{bmatrix}-2\\-6\\2\\-2\end{bmatrix},\begin{bmatrix}0\\1\\-9\\2\end{bmatrix},\begin{bmatrix}2\\5\\-3\\1\end{bmatrix}\right\},$$
where all vectors are in $\mathbb{R}^4$? Justify your response.

7. ($\star$) Show that, for any list $\{v_1,\dots v_n\}$ of vectors in a vector space V, $\operatorname{span}\{v_1,\dots v_n\}$ is the smallest (inclusion-wise) subspace of V containing $\{v_1,\dots v_n\}$.

8. Give an example of a list of vectors in $\mathbb{R}^3$ which is linearly *dependent*. Find some v in the span of the list such that there are two distinct linear combinations of the list equal to v.

9. Let V be a vector space over the field $\mathbb{F}$. Show that every list of vectors in the trivial subspace of V is both linearly independent and spanning (i.e., spans the entire trivial subspace).

10. ($\star$) Show that every spanning list of vectors $L = \{v_1,\dots v_n\}$ in a vector space V can be made into a linearly independent list L' such that $\operatorname{span}L = \operatorname{span}L'$.

Bonus. Let $\mathbb{R}^2$ be viewed as a vector space over the field $\mathbb{Q}$. Is the list

$$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\sqrt{2}\\0\end{bmatrix}\right\}$$

linearly independent?