Solutions to Supplementary Practice Test 1

Christian B. Hughes

BIE-LA1 - Winter 2025

1. Consider the list of vectors $L = \left(\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right)$ in $\mathbb{Z}_5^4$.

1.1. Describe the span of L.

Solution. Before proceeding with Gaussian elimination, what can we say about the span of L given the length (4) of L? Given that the vectors of L are in $\mathbb{Z}_5^4$, and that there are four of them, we cannot conclude anything from this information alone, as they could span a line, a plane, a "3-D space" (hyper-plane), or all of $\mathbb{Z}_5^4$. We will use Gaussian elimination to determine the kind of space these vectors span.

Recall that asking about the span of this list is the same as asking about the structure (and in particular the constraints) of the vectors $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \in \mathbb{Z}_5^4$ satisfying

$$c_1 \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} + c_4 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}.$$

for $c_1, c_2, c_3, c_4 \in \mathbb{Z}_5$. We will simplify this system of equations by applying Gaussian elimination to the matrix with columns consisting of the elements of L, augmented by $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$. We first form the augmented matrix

$$M = \left(\begin{array}{cccc|c} 2 & 1 & 0 & 3 & d_1 \\ 1 & 4 & 2 & 2 & d_2 \\ 3 & 0 & 1 & 1 & d_3 \\ 4 & 2 & 3 & 0 & d_4 \end{array} \right).$$

We now apply Gaussian elimination to M:

$$R_3 \leftarrow R_1 + R_3 : \left(\begin{array}{cccc|c} 2 & 1 & 0 & 3 & d_1 \\ 1 & 4 & 2 & 2 & d_2 \\ 0 & 1 & 1 & 4 & d_1 + d_3 \\ 4 & 2 & 3 & 0 & d_4 \end{array} \right)$$

$$R_4 \leftarrow 3R_1 + R_4 : \left(\begin{array}{cccc|c} 2 & 1 & 0 & 3 & d_1 \\ 1 & 4 & 2 & 2 & d_2 \\ 0 & 1 & 1 & 4 & d_1 + d_3 \\ 0 & 0 & 3 & 4 & 3d_1 + d_4 \end{array}\right)$$

$$R_2 \leftarrow 2R_1 + R_2 : \left(\begin{array}{cccc|c} 2 & 1 & 0 & 3 & d_1 \\ 0 & 1 & 2 & 3 & 2d_1 + d_2 \\ 0 & 1 & 1 & 4 & d_1 + d_3 \\ 0 & 0 & 3 & 4 & 3d_1 + d_4 \end{array}\right)$$

$$R_1 \leftarrow 4R_2 + R_1 : \left(\begin{array}{cccc|c} 2 & 0 & 3 & 0 & 4d_1 + 4d_2 \\ 0 & 1 & 2 & 3 & 2d_1 + d_2 \\ 0 & 1 & 1 & 4 & d_1 + d_3 \\ 0 & 0 & 3 & 4 & 3d_1 + d_4 \end{array}\right)$$

$$R_3 \leftarrow 4R_2 + R_3 : \left(\begin{array}{cccc|c} 2 & 0 & 3 & 0 & 4d_1 + 4d_2 \\ 0 & 1 & 2 & 3 & 2d_1 + d_2 \\ 0 & 0 & 4 & 1 & 4d_1 + 4d_2 + d_3 \\ 0 & 0 & 3 & 4 & 3d_1 + d_4 \end{array}\right)$$

$$R_4 \leftarrow 3R_3 + R_4 : \left(\begin{array}{cccc|c} 2 & 0 & 3 & 0 & 4d_1 + 4d_2 \\ 0 & 1 & 2 & 3 & 2d_1 + d_2 \\ 0 & 0 & 4 & 1 & 4d_1 + 4d_2 + d_3 \\ 0 & 0 & 0 & 2 & 2d_2 + d_3 + d_4 \end{array}\right)$$

The matrix post Gaussian elimination has a pivot in every row. Consequently, L spans $\mathbb{Z}_5^4$.

1.2. Is L linearly independent in $\mathbb{Z}_5^4$?

Solution. From the previous part of this question, we see that L comprises a spanning list of length 4 in $\mathbb{Z}_5^4$. The dimension of $\mathbb{Z}_5^4$ is 4. Given a vector space of dimension n, every spanning list of length n is linearly independent. Thus L is linearly independent in $\mathbb{Z}_5^4$.

1.3. Determine whether L is a basis for $\mathbb{Z}_5^4$

Solution. From the previous two parts of this question, we have that L spans $\mathbb{Z}_5^4$ and is linearly independent. A list of vectors is a basis for a space if and only if it is linearly independent and spans the space. Consequently, L is a basis for $\mathbb{Z}_5^4$.

1.4. Compute the dimension of the span of L.

Solution. Since span$L = \mathbb{Z}_5^4$, and $\mathbb{Z}_5^4$ has dimension 4, span$L$ has dimension 4.

2. Is the set

$$U = \left\{ \left[\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right] \in \mathbb{R}^3 : a_2 = a_1 + 2a_1a_3 + a_3 \right\}.$$

a subspace of $\mathbb{R}^3$?

Solution. Consider the following two elements of U: $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. We have

$$\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}.$$

However, $5 \neq 1 + 2 \cdot 2 \cdot 1 + 1$. Consequently, U fails to be closed under vector addition, and is thus not a subspace of $\mathbb{R}^3$.