Solutions to Supplementary Practice Test 2

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BIE-LA1 - Winter 2025

1. Consider the list of vectors $L = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \\ 17 \end{bmatrix}, \begin{bmatrix} 19 \\ 17 \\ 43 \\ 18 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right)$ in $\mathbb{R}^4$.

   1.1. Describe the span of L.

Solution. We have four vectors in $\mathbb{R}^4$. Thus, it is difficult to immediately conclude anything about the structure of the span of L. We will resort to Gaussian elimination. Recall that asking about the span of this list is the same as asking about the structure (and in particular the constraints) of the vectors $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \in \mathbb{R}^4$ satisfying

$$c_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \\ 17 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 19 \\ 17 \\ 43 \\ 18 \end{bmatrix} + c_4 \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}.$$

for $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Equivalently, this describes a system of linear equations, and we can apply Gaussian elimination to the corresponding matrix

$$M = \left(\begin{array}{cccc|c} 1 & -1 & 19 & 2 & d_1 \\ 2 & 4 & 17 & 1 & d_2 \\ 3 & 1 & 43 & 4 & d_3 \\ 5 & 17 & 18 & -1 & d_4 \end{array} \right).$$

We now apply Gaussian elimination to M over $\mathbb{R}$.

$$R_2 \leftarrow R_2 - 2R_1 : \left(\begin{array}{cccc|c} 1 & -1 & 19 & 2 & d_1 \\ 0 & 6 & -21 & -3 & d_2 - 2d_1 \\ 3 & 1 & 43 & 4 & d_3 \\ 5 & 17 & 18 & -1 & d_4 \end{array} \right)$$

$$R_3 \leftarrow R_3 - 3R_1 : \left(\begin{array}{cccc|c} 1 & -1 & 19 & 2 & d_1 \\ 0 & 6 & -21 & -3 & d_2 - 2d_1 \\ 0 & 4 & -14 & -2 & d_3 - 3d_1 \\ 5 & 17 & 18 & -1 & d_4 \end{array} \right)$$

$$R_4 \leftarrow R_4 - 5R_1 : \left(\begin{array}{cccc|c} 1 & -1 & 19 & 2 & d_1 \\ 0 & 6 & -21 & -3 & d_2 - 2d_1 \\ 0 & 4 & -14 & -2 & d_3 - 3d_1 \\ 0 & 22 & -77 & -11 & d_4 - 5d_1 \end{array}\right)$$

$$R_3 \leftarrow 2R_2 - 3R_3 : \left(\begin{array}{cccc|c} 1 & -1 & 19 & 2 & d_1 \\ 0 & 6 & -21 & -3 & d_2 - 2d_1 \\ 0 & 0 & 0 & 0 & 5d_1 + 2d_2 - 3d_3 \\ 0 & 22 & -77 & -11 & d_4 - 5d_1 \end{array}\right)$$

$$R_4 \leftarrow 11R_2 - 3R_4 : \left(\begin{array}{cccc|c} 1 & -1 & 19 & 2 & d_1 \\ 0 & 6 & -21 & -3 & d_2 - 2d_1 \\ 0 & 0 & 0 & 0 & 5d_1 + 2d_2 - 3d_3 \\ 0 & 0 & 0 & 0 & -7d_1 + 11d_2 - 3d_4 \end{array}\right)$$

Note that the last two rows now correspond to the equations $0 = -7d_1 + 11d_2 - 3d_4$ and $0 = 5d_1 + 2d_2 - 3d_3$. Thus we have two free variables and two constrained variables according to the constraints just given.

Consequently, the set of all vectors in the span of vectors in L is precisely

$$\operatorname{span}(L) = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{r} 5d_1 + 2d_2 - 3d_3 = 0, \\ -7d_1 + 11d_2 - 3d_4 = 0 \end{array} \right\}.$$

1.2. Is L linearly independent in $\mathbb{R}^4$?

Solution. The list L is not linearly independent in $\mathbb{R}^4$, since there are two columns with nonzero entries and no pivots in the row-echelon form of M. Another way to see this is that four vectors in $\mathbb{R}^4$ span $\mathbb{R}^4$ if and only if they are linearly independent in $\mathbb{R}^4$, and since the list L does not span $\mathbb{R}^4$, it cannot be linearly independent in $\mathbb{R}^4$.

1.3. Determine whether L is a basis for $\mathbb{R}^4$

Solution. A list of vectors is a basis for a space if and only if the list is both linearly independent and spans the space. From the previous question, the list of vectors is not linearly independent in $\mathbb{R}^4$, and consequently is not a basis for $\mathbb{R}^4$.

1.4. Compute the dimension of the span of L.

Solution. Observing the row-echelon form of M from the first question, we notice that there are two zero rows and two rows with pivots. Removing the vectors corresponding to the last two columns of the row-reduced matrix M preserves these row pivots, while removing the linearly dependent vectors (the vectors corresponding to the columns with no pivots in their rows). Through this process, via the linear dependence lemma, we obtain a linearly independent list which spans span$L$, and thus we have a basis of length 2 for span$L$. The dimension of span$L$ is thus 2.

2. Is the set

$$U = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in \mathbb{Z}_7^4 : a_1 + a_2 + a_3 + a_4 = 0 \text{ mod} 7 \right\}.$$

a subspace of $\mathbb{Z}_7^4$?

Solution. Consider arbitrary vectors $u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $v = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ in U. We first check that U is closed under vector addition

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 \end{bmatrix}.$$

Clearly $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4$ is 0 modulo 7 when $a_1 + a_2 + a_3 + a_4$ and $b_1 + b_2 + b_3 + b_4$ are 0 modulo 7, so U is closed under vector addition. Now consider an arbitrary scalar $c \in \mathbb{Z}_7$ and an arbitrary vector $v = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ in U. We have

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \\ ca_4 \end{bmatrix}.$$

Evidently, $ca_1 + ca_2 + ca_3 + ca_4 = c(a_1 + a_2 + a_3 + a_4) = c \cdot 0 \text{ mod} 7 = 0 \text{ mod} 7$. Hence, U is closed under vector addition and scalar multiplication, and consequently U is a vector subspace of $\mathbb{Z}_7^4$.